Asymptotic completeness for superradiant Klein-Gordon equations and applications to the De Sitter Kerr metric

Vladimir Georgescu, Christian Gérard, Dietrich Häfner

ASYMPTOTIC COMPLETENESS FOR SUPERRADIANT KLEIN-GORDON EQUATIONS AND APPLICATIONS TO THE DE SITTER KERR METRIC

V. GEORGESCU, C. GÉRARD, AND D. HÄFNER

Abstract. We show asymptotic completeness for a class of superradiant Klein-Gordon equations. Our results are applied to the Klein-Gordon equation on the De Sitter Kerr metric with small angular momentum of the black hole. For this equation we obtain asymptotic completeness for fixed angular momentum of the field.

1. Introduction

1.1. **Introduction.** *Asymptotic completeness* is one of the fundamental properties one might want to show for a Hamiltonian describing the dynamics of a physical system. Roughly speaking it states that the Hamiltonian of the system is equivalent to a free Hamiltonian for which the dynamics is well understood. The dynamics that we want to understand behaves then at large times like this free dynamics modulo possible eigenvalues. In the case when the Hamiltonian is selfadjoint with respect to some suitable Hilbert space inner product an enormous amount of literature has been dedicated to this question. The question is much less studied in the case when the Hamiltonian is not selfadjoint. This situation occurs for example for the Klein-Gordon equation when the field is coupled to a (strong) electric field. This system has been studied by Kako in a short range case (see [23]) and by C. Gérard in the long range case (see [14]). In this situation the Hamiltonian, although not selfadjoint on a Hilbert space, is selfadjoint on a so called *Krein space*. In a previous paper [15] we addressed the question of boundary values of the resolvent for selfadjoint operators on Krein spaces. Applications to propagation estimates for the Klein-Gordon equation are given in [16].

The Klein Gordon equation can be written in a quite general setting in the form

$$(\partial_t^2 - 2ik\partial_t + h)u = 0, \ u : \ \mathbb{R} \to \mathcal{H} \tag{1.1}$$

with selfadjoint operators k and h. However if h is not positive the natural conserved energy for (1.1)

$$\|\partial_t u\|^2 + (hu|u)$$

is not positive and in general no positive conserved energy is available. This happens in particular when the equation is associated to a lorentzian manifold with no global time-like Killing vector field. In this situation natural positive energies can grow in time, and we will loosely speak about *superradiance*, the most famous example being the (De Sitter) Kerr metric which describes rotating black holes. T his example doesn't enter into the framework of our previous papers because the Hamiltonian can no longer be realized as a selfadjoint operator

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on a Krein space whose topology is given by some natural positive (but not conserved) energy. The problem comes from the fact that the operator k has different "limit operators" in the different ends of the manifold. In the one dimensional case scattering results for this situation have been obtained by Bachelot in [4].

Asymptotic completeness for wave equations on Lorentzian manifolds has been studied for a long time since the works of Dimock and Kay in the 1980's, see e.g. [8]. The main motivation came from the *Hawking effect*. Such results are a necessary step to give mathematically rigorous descriptions of the Hawking effect, see Bachelot [3] and Häfner [20]. The most complete scattering results exist in the Schwarzschild metric, see e.g. Bachelot [2]. Asymptotic completeness has also been shown on the Kerr metric for non superradiant modes of the Klein-Gordon (see Häfner [19]) and for the Dirac equation (for which no superradiance occurs), see Häfner-Nicolas [21]. In this setting asymptotic completeness can be understood as an existence and uniqueness result for the characteristic Cauchy problem in energy space at null infinity, see [21] for details. As far as we are aware asymptotic completeness has not been addressed in the setting of superradiant equations on the (De Sitter) Kerr background. Note however that scattering results have been obtained by Dafermos, Holzegel, Rodnianski in the difficult nonlinear setting of the Einstein equations supposing exponential decay for the scattering data on the future event horizon and on future null infinity, see [7]. Also there has been enormous progress in the last years on a somewhat related question which is the question of decay of the local energy for the wave equation on the (De Sitter) Kerr metric. We mention in this context the papers of Andersson-Blue [1], Dyatlov [10], Dafermos-Rodnianski [6], Finster-Kamran-Smoller-Yau [12], [13], Tataru-Tohaneanu [27] and Vasy [28] as well as references therein for an overview. Let us make some comments on the similarities and differences between asymptotic completeness results and decay of the local energy:

- for a hyperbolic equation like the wave equation, the essential ingredients for asymptotic completeness are *minimal velocity estimates* stating that the energy in cones inside the light cone goes to zero. No precise rate is required, but no loss of derivatives is permitted in the estimates.
- *energy estimates* are necessary for asymptotic completeness. One needs to estimate the energy at null infinity by the energy on a $t = 0$ slice and the energy at the time $t = 0$ slice by the energy at null infinity. Leaving aside the question of loss of derivatives the first estimate can probably be deduced from the local energy decay estimates. For the other direction however a new argument is needed. Indeed most of the local energy decay estimates use the redshift, that becomes a blueshift in the inverse sens of time, see [7].
- the choice of coordinates has probably to be different. Whereas coordinates that extend smoothly across the event horizon are well adapted to the question of decay of local energy, they don't seem to be well adapted for showing asymptotic completeness results.

We refer to [25] for a more detailed discussion on the link between local energy decay and asymptotic completeness results.

We show in this paper asymptotic completeness results for the superradiant Klein-Gordon equation in a quite general setting. Our abstract Klein-Gordon operators have to be understood as operators acting on $\mathbb{R}_t \times \Sigma$, where Σ is a manifold with two ends, both being asymptotically hyperbolic. We also impose the existence of "limit operators" in the ends which can be realized as selfadjoint operators on a Hilbert space. In this setting the non real spectrum of the Hamiltonian consists of a finite number of complex eigenvalues with finite multiplicity, we can define a smooth functional calculus for the Hamiltonian and the truncated resolvent can be extended meromorphically across the real axis. We show propagation estimates for initial data which in energy are supported outside so called singular points. These singular points are closely related to real resonances but unlike the selfadjoint setting they may be singular points which are not real resonances.

From the propagation estimates it follows in particular that the evolution is uniformly bounded for data supported in energy outside the singular points. The same holds true for high energy data for which no superradiance appears.

We apply then our results to the De Sitter Kerr metric with small angular momentum. We show asymptotic completeness for fixed but arbitrary angular momentum n of the field. For angular momentum $n \neq 0$ the absence of real resonances follows from the results of Dyatlov, see [9]. However some additional work is required to show the absence of singular points. In a subsequent paper we will discuss the De Sitter Kerr case in more detail.

1.2. **Abstract Klein-Gordon equation.** Let $(\mathcal{H}, (\cdot|\cdot))$ be a Hilbert space, $(h, D(h))$ be a selfadjoint operator on $\mathcal{H}$ and $k \in \mathcal{B}(\langle h\rangle^{-1/2}\mathcal{H};\ \langle h\rangle^{1/2}\mathcal{H})$ be another selfadjoint operator. We consider the following abstract Klein-Gordon equation :

$$\left\{\begin{array}{rcl} (\partial_t^2 - 2ik\partial_t + h)u &=& 0, \\ u|_{t=0} &=& u_0, \\ \partial_t u|_{t=0} &=& u_1. \end{array}\right. \tag{1.2}$$

The operator h will be in general not positive. As a consequence the conserved *energy*

$$\langle u, u\rangle_0 := \|\partial_t u\|^2_{\mathcal{H}} + (hu|u)_{\mathcal{H}}$$

will in general not be positive. The hyperbolic character of the equation is expressed by the condition

$$h_0 := h + k^2 \geq 0.$$

Another important conserved quantity is the *charge* which is given by

$$q(u, v) = \langle u|v\rangle = (u_0|v_1) + (u_1|v_0)$$

Setting $\Psi_0 = (u_0, -iu_1)$ we can rewrite (1.2) as a first order equation

$$\left\{\begin{array}{rcl} (\partial_t - iH)\Psi &=& 0, \\ \Psi|_{t=0} &=& \Psi_0, \end{array}\right. \tag{1.3}$$

where

$$H = \begin{pmatrix} 0 & \mathbb{1} \\ h & 2k \end{pmatrix}.$$

Observe that

$$(H - z)^{-1} = p(z)^{-1} \begin{pmatrix} z - 2k & 1 \\ h & z \end{pmatrix}, \tag{1.4}$$

where

$$p(z) = h + z(2k - z) = h_0 - (k - z)^2 \in B(\langle h_0\rangle^{-1/2}\mathcal{H}, \langle h_0\rangle^{1/2}\mathcal{H}),\ z \in \mathbb{C}.$$

The map $z \mapsto p(z)$ is called a *quadratic pencil* and plays an important role in the study of H.

1.3. **Results for the Klein-Gordon equation on the De Sitter Kerr metric.** Let $(\mathcal{M}, g)$ be the De Sitter Kerr spacetime and u a solution of the Klein-Gordon equation

$$\left\{\begin{array}{rcl}(\Box_g + m^2)u &=& 0,\\ u|_{t=0} &=& u_0,\\ \partial_t u|_{t=0} &=& u_1,\end{array}\right. \tag{1.5}$$

where t is the Boyer-Lindquist time. We define the Cauchy surface $\Sigma = \{t = 0\}$. We write (1.5) in the form (1.2) and associate the homogeneous energy space $\dot{\mathcal{E}}$, which is the completion of $C_0^\infty(\Sigma) \times C_0^\infty(\Sigma)$ for the norm

$$\|u\|_{\dot{\mathcal{E}}} = \|u_1 - ku_0\|_{\mathcal{H}}^2 + ((h + k^2)u_0|u_0)_{\mathcal{H}},$$

where $\mathcal{H} = L^2(\Sigma; dVol)$ for a suitable measure $dVol$. We can take $dVol = dxd\omega$, where x is a Regge-Wheeler type coordinate if we multiply u with a suitable function. We then write the equation in the form (1.3) and obtain a Hamiltonian $\dot{H}$ (formally $\dot{H} = H$) acting on $\dot{\mathcal{E}}$. We also consider the wave equations

$$\left\{\begin{array}{rcl}(\partial_t^2 - 2\Omega_{l/r}\partial_\varphi\partial_t - \partial_x^2 + \Omega_{l/r}^2\partial_\varphi^2)u^{l/r} &=& 0,\\ u^{l/r}|_{t=0} &=& u_0^{l/r},\\ \partial_t u^{l/r}|_{t=0} &=& u_1^{l/r},\end{array}\right. \tag{1.6}$$

where $\Omega_{l/r}$ is the angular velocity of the black hole resp. cosmological horizon. Let $H_{l/r}$, $\dot{\mathcal{E}}_{l/r}$ be the associated Hamiltonians and homogeneous energy spaces. We denote by $\dot{\mathcal{E}}^n$, $\dot{\mathcal{E}}^n_{l/r}$ the subspaces of angular momentum n. Let $i_{l/r}$ be smooth cut-off functions equal to one at $\mp\infty$, equal to zero at $\pm\infty$. Let a be the angular momentum per unit mass of the De Sitter Kerr spacetime. The main result in the De Sitter Kerr setting is the following

Theorem 1.1. *There exists $a_0 > 0$ such that for all $|a| < a_0$ and $n \in \mathbb{Z} \setminus \{0\}$ we have the following.*

i) There exists a dense subspace $\mathcal{E}^{fin,n}_{l/r}$ of $\dot{\mathcal{E}}^n_{l/r}$ such that for all $v_{l/r} \in \mathcal{E}^{fin,n}_{l/r}$ the limits

$$W_\pm^{l/r} v_{l/r} := \lim_{t\to\pm\infty} e^{it\dot{H}} i_{l/r} e^{-it\dot{H}_{l/r}} v_{l/r}$$

exist. The operators $W_\pm^{l/r}$ extend to bounded operators $W_\pm^{l/r} \in \mathcal{B}(\dot{\mathcal{E}}^n_{l/r}; \dot{\mathcal{E}}^n)$.

ii) For all $u \in \dot{\mathcal{E}}^n$ the limit

$$\Omega_\pm^{l/r} u := \lim_{t\to\pm\infty} e^{it\dot{H}_{l/r}} i_{l/r} e^{-it\dot{H}} u$$

exists in $\dot{\mathcal{E}}^n_{l/r}$.

i), ii) also hold for $n = 0$ if the mass m of the field is strictly positive.

1.4. **Plan of the paper.**

– in Sect. 2 we collect some results on general abstract Klein-Gordon equations. Also we give some basic resolvent estimates. It turns out that already in this abstract setting superradiance can only occur at low frequencies as expressed in the estimates of Lemma 2.6. This fact is already known for the Kerr metric, see e.g. Dafermos Rodnianski [6], but not in this spectral formulation. We also study *gauge transformations*

in Sect. 2.5.3. These gauge transformations correspond to choices of different Killing fields in a more geometric language.
- in Sect. 3 we recall some facts on meromorphic Fredholm theory. We show that a meromorphic extension of the truncated resolvent of h gives a meromorphic extension of the weighted resolvent of H.
- in Sect. 4 we describe the abstract setting for a Klein-Gordon operator on a manifold with two ends. Our assumptions assure that the asymptotic Hamiltonians in the ends are selfadjoint. Gluing the resolvents of the asymptotic operators together gives the resolvent for H using the Fredholm theory of Sect. 3. We obtain in this way resolvent estimates for the Hamiltonian H which are sufficient to construct a smooth functional calculus for H.
- in Sect. 5 we prove propagation estimates which are needed for the proof of the asymptotic completeness result. We also introduce the notion of *singular points*. Singular points are obstacles for uniform boundedness of the evolution and therefore also for asymptotic completeness. A useful criterion for the absence of singular points is given.
- in Sect. 6 we show uniform boundedness of the evolution for data which is spectrally supported outside the singular points.
- asymptotic completeness is shown in the abstract setting in Sect. 7. The scattering space corresponds to data which are supported in energy outside the singular points.
- in Sect. 8 we introduce the geometric setting. Our operators fulfill the hypotheses of the abstract setting. We obtain meromorphic extensions of the weighted resolvents using a result of Mazzeo-Melrose [24].
- We apply our general result to the geometric setting in Section 9 and obtain an asymptotic completeness result in the geometric setting.
- in Sect. 10 we describe the Klein-Gordon equation on the De Sitter Kerr metric.
- in Sect. 11 the main results are formulated in the De Sitter Kerr setting. Two types of results are established: comparison to spherically symmetric asymptotic dynamics on the same energy space and comparison to asymptotic profiles. These asymptotic profiles give rise to energy spaces which are bigger than the original ones. The wave operators can therefore only be defined as limits on dense subspaces. They then extend by continuity to the whole energy space for the profiles. Inverse wave operators exist on the whole energy space as limits.
- The proofs of the theorems in the De Sitter Kerr setting are given in Sect. 12. We apply our earlier abstract theorems. To obtain the meromorphic extensions of the different truncated resolvents it is crucial that the cosmological constant is strictly positive. The absence of real resonances and complex eigenvalues follows from the work of Dyatlov [9] for a compactly supported cutoff resolvent. An hypo-ellipticity argument enables us to use an exponential weight. Our earlier results in [16] and the general criterion in Section 5 enable us to show the absence of singular points.

As usual for asymptotic completeness results and to simplify the exposition we consider only the limit $t \to \infty$. All the results in this paper also hold in the limit $t \to -\infty$ and the proofs are the same.

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2. Background on abstract Klein-Gordon operators

2.1. Notations.

- if X, Y are sets and $f : X \to Y$, we write $f : X \tilde{\to} Y$ if f is bijective. We use the same notation if X, Y are topological spaces if f is an isomorphism.
- if $\mathcal{H}$ is a Banach space we denote $\mathcal{H}^*$ its adjoint space, the set of continuous anti-linear functionals on $\mathcal{H}$ equipped with the natural Banach space structure. Thus the canonical anti-duality $\langle u, w\rangle$, where $u \in \mathcal{H}$ and $w \in \mathcal{H}^*$, is anti-linear in u and linear in w. In general we denote by $\langle \cdot|\cdot\rangle$ hermitian forms on $\mathcal{H}$, again anti-linear in the first argument and linear in the second one, but if $\mathcal{H}$ is a Hilbert space its scalar product is denoted by $(\cdot|\cdot)$.
- $\mathcal{B}(\mathcal{H})$ is the space of bounded operators on $\mathcal{H}$ and $\mathcal{B}_\infty(\mathcal{H})$ the subspace of compact operators.
- if S is a closed densely defined operator on a Banach space, then $D(S)$, $\rho(S)$, $\sigma(S)$ are its domain, resolvent set and spectrum. We use the notation $\langle S\rangle = (1 + S^2)^{1/2}$ if S is an operator for which this expression has a meaning, in particular if S is a real number.
- if S is a self-adjoint operator on a Hilbert space then $S > 0$ means $S \geq 0$ and $\operatorname{Ker} S = \{0\}$.
- if A, B are two selfadjoint operators or real numbers, (possibly depending on some parameters), we write $A \lesssim B$ if $A \leq CB$ for some constant $C > 0$, (uniformly with respect to the parameters).

2.2. Scales of Hilbert spaces.

Let $\mathcal{H}$ be a Hilbert space identified with its adjoint space $\mathcal{H}^* = \mathcal{H}$ via the Riesz isomorphism. If h is a selfadjoint operator on $\mathcal{H}$ we associate to it the *non-homogeneous Sobolev spaces*

$$\langle h\rangle^{-s}\mathcal{H} := \mathrm{Dom}|h|^s, \ \langle h\rangle^{s}\mathcal{H} := (\langle h\rangle^{-s}\mathcal{H})^*, \ s \geq 0.$$

The spaces $\langle h\rangle^{-s}\mathcal{H}$ are equipped with the graph norm $\|\langle h\rangle^s u\|$. We keep the notation

$$(u|v), \ u \in \langle h\rangle^{-s}\mathcal{H}, \ v \in \langle h\rangle^{s}\mathcal{H},$$

to denote the duality bracket between $\langle h\rangle^{-s}\mathcal{H}$ and $\langle h\rangle^{s}\mathcal{H}$.

If $\operatorname{Ker} h = \{0\}$ then we also define the *homogeneous Sobolev space* $|h|^s\mathcal{H}$ equal to the completion of $\mathrm{Dom}|h|^{-s}$ for the norm $\||h|^{-s}u\|$. The notation $\langle h\rangle^s\mathcal{H}$ or $|h|^s\mathcal{H}$ is convenient but somewhat ambiguous because usually $a\mathcal{H}$ denotes the image of $\mathcal{H}$ under the linear operator a. We refer to [16, Subsect. 2.1] for a complete discussion of this question.

Let us mention some properties of the scales of spaces defined above:

$$\langle h\rangle^{-s}\mathcal{H}\subset\langle h\rangle^{-t}\mathcal{H},\ \text{if } t\le s,\ \langle h\rangle^{-s}\mathcal{H}\subset|h|^{-s}\mathcal{H}, |h|^{s}\mathcal{H}\subset\langle h\rangle^{s}\mathcal{H} \text{ if } s\ge 0,$$
$$\langle h\rangle^{0}\mathcal{H}=|h|^{0}\mathcal{H}=\mathcal{H},\ \langle h\rangle^{s}\mathcal{H}=(\langle h\rangle^{-s}\mathcal{H})^*,\ |h|^{s}\mathcal{H}=(|h|^{-s}\mathcal{H})^*,$$
$$0\in\rho(h)\Leftrightarrow\ \langle h\rangle^{s}\mathcal{H}=|h|^{s}\mathcal{H} \text{ for some } s\neq 0\Leftrightarrow\langle h\rangle^{s}\mathcal{H}=|h|^{s}\mathcal{H} \text{ for all } s,$$
$$\text{the operator } |h|^s \text{ is unitary from } |h|^{-t}\mathcal{H} \text{ to } |h|^{s-t}\mathcal{H} \text{ for all } s,t\in\mathbb{R}.$$

2.3. **Quadratic pencils.** Let $\mathcal{H}$ be a Hilbert space, h a selfadjoint operator on $\mathcal{H}$, and $k\in B(\mathcal{H})$ a *bounded* symmetric operator. Then $h_0=h+k^2$ is a self-adjoint operator on $\mathcal{H}$ with the same domain as h hence $\langle h\rangle^s\mathcal{H}=\langle h_0\rangle^s\mathcal{H}$ for $s\in[-1,1]$. Thus the operators h and h_0 define the same scale of Sobolev spaces for $s\in[-1,1]$ that we shall denote:

$$\mathcal{H}^s:=\langle h\rangle^{-s}\mathcal{H}=\langle h_0\rangle^{-s}\mathcal{H}\quad\text{if } -1\le s\le 1.$$

We define the *quadratic pencil*

$$p(z)=h+z(2k-z)=h_0-(k-z)^2,\ z\in\mathbb{C}.$$

A priori these are operators on $\mathcal{H}$ with domain $\mathcal{H}^1$ and we clearly have $p(z)^*=p(\overline{z})$ as operators on $\mathcal{H}$. Moreover, for each $s\in[0,1]$ they extend to operators in $B(\mathcal{H}^s,\mathcal{H}^{s-1})$ and, for example, the relation $p(z)^*=p(\overline{z})$ holds as operators $\mathcal{H}^{\frac12}\to\mathcal{H}^{-\frac12}$. From this it is easy to deduce the following lemma (see [15, Lemma 8.1] for a more general result).

Lemma 2.1. *The following conditions are equivalent:*

$$\begin{array}{ll}(1)\ p(z):\mathcal{H}^1\tilde{\to}\mathcal{H} & (2)\ p(\overline{z}):\mathcal{H}^1\tilde{\to}\mathcal{H}\\ (3)\ p(z):\mathcal{H}^{-\frac12}\tilde{\to}\mathcal{H}^{\frac12} & (4)\ p(\overline{z}):\mathcal{H}^{-\frac12}\tilde{\to}\mathcal{H}^{\frac12}\\ (5)\ p(z):\mathcal{H}\tilde{\to}\mathcal{H}^{-1} & (6)\ p(\overline{z}):\mathcal{H}\tilde{\to}\mathcal{H}^{-1}.\end{array}$$

In particular, the set

$$\rho(h,k):=\{z\in\mathbb{C}\mid p(z):\mathcal{H}^{-\frac12}\tilde{\to}\mathcal{H}^{\frac12}\}=\{z\in\mathbb{C}\mid p(z):\mathcal{H}^1\tilde{\to}\mathcal{H}\}\tag{2.1}$$

is invariant under conjugation.

The next result is easy to prove in the present context; one can find a proof under more general conditions in [15, Lemma 8.2].

Proposition 2.2. *If h is bounded below then there exists $c_0>0$ such that*

$$\{z\ :|\mathrm{Im}z|>|\mathrm{Re}z|+c_0\}\subset\rho(h,k).$$

We shall prove now some estimates on $p(z)^{-1}$ for $z\in\rho(h,k)$. Note that they are valid under much more general assumptions on h and k then those imposed in this paper.

Lemma 2.3. *Assume that $h+c\ge 0$ for some $0\le c$ and let $b>1$. If $z\in\rho(h,k)$ then*

$$\|p(z)^{-1}\|\le\frac{b}{|z\,\mathrm{Im}z|}\quad\textit{if}\quad|z|^2\ge\frac{bc}{b-1}.\tag{2.2}$$

Proof. We abbreviate $p=p(z)$ and $\mu=\mathrm{Im}z$. The main point is the identity

$$\mathrm{Im}\frac{z}{\mu p}=\frac{1}{p^*}(h+|z|^2)\frac{1}{p}\tag{2.3}$$

which is rather obvious:

$$\frac{z}{p}-\frac{\overline{z}}{p^*}=\frac{1}{p^*}\big(zp^*-\overline{z}p\big)\frac{1}{p}=(z-\overline{z})\frac{1}{p^*}\big(h+|z|^2\big)\frac{1}{p}.$$

Then the relation (2.3) gives $(|z|^2-c)\frac{1}{p^*}\frac{1}{p}\leq \mathrm{Im}\frac{z}{\mu p}$ hence

$$|\mu|(|z|^2-c)\|p^{-1}u\|^2\leq \big|\mathrm{Im}(u|zp^{-1}u)\big|\leq |z|\|u\|\|p^{-1}u\|$$

hence $|\mu|(|z|^2-c)\|p^{-1}u\|^2\leq |z|\|u\|$ which is more than required. □

Lemma 2.4. *Assume $h_0\geq 0$ and $k^2\leq \alpha h_0+\beta$ with $\alpha<1$. Then h is bounded from below and if $h+c\geq 0$ and $\varepsilon>0$ then there is a number C such that for $z\in\rho(h,k)$ and $|z|\geq\sqrt{c}+\varepsilon$*

$$\|h_0^{\frac{1}{2}}p(z)^{-1}\|\leq C|\mathrm{Im}z|^{-1}. \tag{2.4}$$

Proof. If we set $q=p^{-1}$ then (2.3) implies $q^*\big((1-\alpha)h_0+|z|^2\big)q\leq\beta q^*q+\mu^{-1}\mathrm{Im}\,zq$ hence

$$(1-\alpha)\|h_0^{\frac{1}{2}}p^{-1}u\|^2\leq\beta\|p^{-1}u\|^2+|z/\mu|\|u\|\|p^{-1}u\|\leq\tfrac{\beta b^2}{|z\mu|^2}\|u\|^2+\tfrac{b}{\mu^2}\|u\|^2$$

if $|z|^2\geq\frac{bc}{b-1}$. This estimate is more precise than (2.4). Note that we may take $c=\beta$. □

2.4. **Spaces.** The operators h,k,h_0 and the spaces $\mathcal{H}^s$ are as in the preceding subsection, in particular k is a bounded operator in $\mathcal{H}$, but now we shall impose much stronger conditions.

From now on *we always assume* that the following condition is satisfied:

$$h_0:=h+k^2>0. \tag{A1}$$

Then the homogeneous scale $h_0^s\mathcal{H}$ associated to h_0 is well defined. Note that, if h is injective, the spaces $|h|^{-1}\mathcal{H}$ and $h_0^{-1}\mathcal{H}$ are quite different in general, although $\langle h\rangle^{-1}\mathcal{H}=\langle h_0\rangle^{-1}\mathcal{H}$.

We shall require k to behave well with respect to the homogeneous h_0-scale:

$$\begin{cases} k\in\mathcal{B}(h_0^{-1/2}\mathcal{H});\\ \text{if } z\notin\mathbb{R} \text{ then } (k-z)^{-1}\in\mathcal{B}(h_0^{-1/2}\mathcal{H}) \text{ and}\\ \|(k-z)^{-1}\|_{\mathcal{B}(h_0^{-1/2}\mathcal{H})}\lesssim|\mathrm{Im}z|^{-n} \text{ for some } n>0;\\ \exists m>0 \text{ such that if } |z|\geq m\|k\|_{\mathcal{B}(\mathcal{H})} \text{ then}\\ \|(k-z)^{-1}\|_{\mathcal{B}(h_0^{-1/2}\mathcal{H})}\lesssim\big||z|-\|k\|_{\mathcal{B}(\mathcal{H})}\big|^{-1}. \end{cases} \tag{A2}$$

The next comments will clarify the meaning of these conditions. Recall that $h_0^{-s}\mathcal{H}$ and $h_0^s\mathcal{H}$ are adjoints to each other but they are not comparable and neither are they comparable with $\mathcal{H}$. The first assumption says that the operator k leaves $D(h_0^{1/2})$ invariant and that its restriction to $D(h_0^{1/2})$ extends to a bounded operator, say $\overline{k}$, in $h_0^{-1/2}\mathcal{H}$. The rest of the assumption concerns the resolvent of $\overline{k}$ in this space. In order not to overcharge the notation we kept the notation k for $\overline{k}$.

The preceding assumptions allow us to get a new estimate on the quadratic pencil p.

Lemma 2.5. *Under the conditions* (A1) *and* (A2)*, there are numbers $C,M>0$ such that*

$$\|h_0^{\frac{1}{2}}p(z)^{-1}(k-z)u\|\leq C|\mathrm{Im}z|^{-1}\|h_0^{\frac{1}{2}}u\|\quad \textit{if } |z|\geq M\|k\|_{\mathcal{B}(\mathcal{H})}. \tag{2.5}$$

Proof. We abbreviate $p = p(z)$ and $m = z - k$, so that $m^* = \overline{z} - k$ and $p = h_0 - m^2$. We have:

$$\frac{z}{m}h_0\frac{1}{p}m - m^*\frac{1}{p^*}h_0\frac{\overline{z}}{m^*} = m^*\frac{1}{p^*}\left(p^*\frac{z}{|m|^2}h_0 - h_0\frac{\overline{z}}{|m|^2}p\right)\frac{1}{p}m.$$

If we replace here p by $h_0 - m^2$ and then develop and rearrange the terms, we get:

$$m^*\frac{1}{p^*}\left((z-\overline{z})h_0\frac{1}{|m|^2}h_0 + h_0\frac{\overline{z}m}{m^*} - \frac{zm^*}{m}h_0\right)\frac{1}{p}m.$$

Since $\frac{m^*}{m} = 1 - \frac{z-\overline{z}}{m}$ and $\frac{z}{m} = 1 + \frac{k}{m}$, a simple computation gives:

$$h_0\frac{\overline{z}m}{m^*} - \frac{zm^*}{m}h_0 = (z-\overline{z})\left(h_0 + h_0\frac{k}{m^*} + \frac{k}{m}h_0\right).$$

To conclude, we have proved, with $\mu = \mathrm{Im}z$,

$$\frac{1}{\mu}\mathrm{Im}\left(\frac{z}{m}h_0\frac{1}{p}m\right) = m^*\frac{1}{p^*}\left(h_0 + 2\mathrm{Re}(\frac{k}{m}h_0) + h_0\frac{1}{|m|^2}h_0\right)\frac{1}{p}m.$$

We may also write this as follows:

$$\frac{1}{\mu}\mathrm{Im}(u|zm^{-1}h_0p^{-1}mu) = \|h_0^{\frac{1}{2}}p^{-1}mu\|^2 + 2\mathrm{Re}(p^{-1}mu|km^{-1}h_0p^{-1}mu) + \|m^{-1}h_0p^{-1}mu\|^2.$$

Since the last term is positive, we get

$$\begin{aligned} rl\|h_0^{\frac{1}{2}}p^{-1}mu\|^2 \leq& \frac{1}{\mu}\mathrm{Im}(u|zm^{-1}h_0p^{-1}mu) - 2\mathrm{Re}(p^{-1}mu|km^{-1}h_0p^{-1}mu) \\ =& \frac{1}{\mu}\mathrm{Im}(h_0^{\frac{1}{2}}u|h_0^{-\frac{1}{2}}zm^{-1}h_0^{\frac{1}{2}}\cdot h_0^{\frac{1}{2}}p^{-1}mu) \\ &- 2\mathrm{Re}(h_0^{\frac{1}{2}}p^{-1}mu|h_0^{-\frac{1}{2}}km^{-1}h_0^{\frac{1}{2}}\cdot h_0^{\frac{1}{2}}p^{-1}mu). \end{aligned}$$

Set $a(z) = \|h_0^{-\frac{1}{2}}zm^{-1}h_0^{\frac{1}{2}}\|$ and $b(z) = 2\|h_0^{-\frac{1}{2}}km^{-1}h_0^{\frac{1}{2}}\|$. Since $zm^{-1} = 1 + km^{-1}$, assumption (A2) implies the boundedness of $a(z)$ for large z and $b(z) \to 0$ if $z \to \infty$. Finally, we have

$$\big(1 - b(z)\big)\|h_0^{\frac{1}{2}}p^{-1}mu\| \leq a(z)|\mu|^{-1}\|h_0^{\frac{1}{2}}u\|,$$

which proves the lemma. □

For easier reference later on, we summarize in the next proposition a particular case of the estimates we got in Lemmas 2.3, 2.4 and 2.5.

Proposition 2.6. *Assume that the conditions* (A1) *and* (A2) *are satisfied and let* $\varepsilon > 0$. *Then there are numbers* $C, M > 0$ *such that:*

$$\|p^{-1}(z)\| \leq C|z|^{-1}|\mathrm{Im}z|^{-1} \ \textit{if}\ |z| \geq (1+\epsilon)\|k\|_{\mathcal{B}(\mathcal{H})}, \tag{2.6}$$

$$\|h_0^{1/2}p^{-1}(z)\| \leq C|\mathrm{Im}z|^{-1} \ \textit{if}\ |z| \geq (1+\epsilon)\|k\|_{\mathcal{B}(\mathcal{H})}, \tag{2.7}$$

$$\|h_0^{1/2}p^{-1}(z)(k-z)u\| \leq C|\mathrm{Im}z|^{-1}\|h_0^{1/2}u\| \ \textit{if}\ |z| \geq M\|k\|_{\mathcal{B}(\mathcal{H})}. \tag{2.8}$$

Sometimes it is useful to consider also the homogeneous h-scale. The following assumption will be convenient in such situations.

(A3) $$h \geq ck^2 \text{ for some real } c > 0.$$

This means that h is positive and $\|ku\| \leq c^{-1/2}\|h^{1/2}u\|$ for all $u \in D(h^{1/2})$.

Lemma 2.7. *If $c > 0$ is real then $h \geq ck^2 \Leftrightarrow h \geq \frac{c}{1+c}h_0$. Thus* (A3) *is satisfied if and only if there is $b > 0$ real such that $bh_0 \leq h \leq h_0$. If* (A1) *and* (A3) *hold then $h > 0$.*

Proof. Note that $h = h_0 - k^2 \leq h_0$. On the other hand, if $c > 0$ is real then we clearly have

$$h \geq ck^2 \Leftrightarrow h_0 \geq (1+c)k^2 \Leftrightarrow h \geq \frac{c}{1+c}h_0 \tag{2.9}$$

and this implies the assertions of the lemma. □

Corollary 2.8. *If* (A1) *and* (A3) *are satisfied then $h_0^s\mathcal{H} = h^s\mathcal{H}$ for all $-1/2 \leq s \leq 1/2$.*

Proof. Indeed, from $bh_0 \leq h \leq h_0$ we get $b^\theta h_0^\theta \leq h^\theta \leq h_0^\theta$ if $0 \leq \theta \leq 1$. □

2.4.1. *Inhomogeneous energy spaces.* The *inhomogeneous* energy space is the vector space

$$\mathcal{E} = \mathcal{H}^{1/2} \oplus \mathcal{H},$$

equipped with the natural direct sum topology which makes it a Hilbertizable space. For consistency with the norm that we introduce later in the homogeneous case we take

$$\|\left(\begin{smallmatrix} u_0 \\ u_1 \end{smallmatrix}\right)\|_{\mathcal{E}}^2 = \|u_1 - ku_0\|^2 + ((h_0+1)u_0|u_0) \tag{2.10}$$

but of course we could replace here k by zero. It is convenient, as explained in [15], to identify its adjoint space $\mathcal{E}^*$ with $\mathcal{H} \oplus \mathcal{H}^{-1/2}$ the anti-duality being given by

$$\langle u, v\rangle = (u_0|v_1 - kv_0) + (u_1 - ku_0|v_0) \quad \text{if} \quad u = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in \mathcal{E}, \quad v = \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \in \mathcal{E}^* \tag{2.11}$$

usually called the *charge*. Observe that $\mathcal{E} \subset \mathcal{E}^*$ continuously and densely. We identify $\mathcal{E}^{**} = \mathcal{E}$ as in the Hilbert space case by setting $\langle v, u\rangle = \overline{\langle u, v\rangle}$.

In what follows it will often be convenient to use the operator

$$\Phi(k) = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}. \tag{2.12}$$

Note that $\Phi(k) : \mathcal{E}\tilde{\rightarrow}\mathcal{E}$ and $\Phi(k) : \mathcal{E}^*\tilde{\rightarrow}\mathcal{E}^*$ with $\Phi^{-1}(k) = \Phi(-k)$ and we may write

$$\mathcal{E} = \Phi(k)(\langle h_0\rangle^{-1/2}\mathcal{H} \oplus \mathcal{H}) \quad \text{and} \quad \mathcal{E}^* = \Phi(k)(\mathcal{H} \oplus \langle h_0\rangle^{1/2}\mathcal{H}), \tag{2.13}$$

which explains the choice of the norm in (2.10) and makes the connection with (2.15).

2.4.2. *Homogeneous energy spaces.* We define the *homogeneous energy space* $\dot{\mathcal{E}}$ as the completion of $\mathcal{E}$ under the norm defined by:

$$\|\left(\begin{smallmatrix} u_0 \\ u_1 \end{smallmatrix}\right)\|_{\dot{\mathcal{E}}}^2 := \|u_1 - ku_0\|^2 + (h_0u_0|u_0). \tag{2.14}$$

The completion is the set of couples $u = \left(\begin{smallmatrix} u_0 \\ u_1 \end{smallmatrix}\right)$ with $u_0 \in h_0^{-1/2}\mathcal{H}, u_1 \in (1 + h_0^{-1/2})\mathcal{H}$, and such that $u_1 - ku_0 \in \mathcal{H}$. We shall realize its adjoint space $\dot{\mathcal{E}}^*$ with the help of the charge anti-duality defined as in (2.11). Observe that, since $k \in \mathcal{B}(h_0^{-1/2}\mathcal{H})$ by (A2), we also have

$$\dot{\mathcal{E}} = \Phi(k)(h_0^{-1/2}\mathcal{H} \oplus \mathcal{H}), \quad \dot{\mathcal{E}}^* = \Phi(k)(\mathcal{H} \oplus h_0^{1/2}\mathcal{H}). \tag{2.15}$$

If the assumption (A3) is satisfied then we also define the h-homogeneous energy spaces

$$\dot{\tilde{\mathcal{E}}} = h^{-1/2}\mathcal{H} \oplus \mathcal{H}, \quad \dot{\tilde{\mathcal{E}}}^* = \mathcal{H} \oplus h^{1/2}\mathcal{H}.$$

Here the direct sums are in the Hilbert space sense and the identification of $\dot{\tilde{\mathcal{E}}}^*$ with the space adjoint to $\dot{\tilde{\mathcal{E}}}$ is done with the help of the sesquilinear form defined as in (2.11) but with $k = 0$.

Lemma 2.9. *Assume* (A1)-(A3). *Then* $\dot{\mathcal{E}} = \dot{\tilde{\mathcal{E}}}$ *and the norms* $\|\cdot\|_{\dot{\mathcal{E}}}$ *and* $\|\cdot\|_{\dot{\tilde{\mathcal{E}}}}$ *are equivalent.*

Proof. We have to prove that $\|u_1 - ku_0\|^2 + (h_0u_0|u_0) \simeq \|u_1\|^2 + (hu_0|u_0)$. But this is obvious by (A3) and Lemma 2.7. □

2.4.3. *Conserved quantities.* On $\mathcal{E}$ we introduce for $\ell \in \mathbb{R}$ the following sesquilinear forms :

$$\langle u|v\rangle_\ell := (u_1 - \ell u_0|v_1 - \ell v_0) + (p(\ell)u_0|v_0), \tag{2.16}$$

where $p(\ell) = h_0 - (k-\ell)^2$. We have seen in the introduction that these forms are formally conserved by the evolution, but in general not positive definite.

Lemma 2.10. *For all* $\ell \in \mathbb{R}$, $\langle\cdot|\cdot\rangle_\ell$ *is continuous with respect to the norm* $\|\cdot\|_{\mathcal{E}}$.

Proof. Due to the polarization identity it suffices to show $|\langle u|u\rangle_\ell| \lesssim \|u\|^2_{\mathcal{E}}$ for all $u \in \mathcal{E}$. Since

$$\langle u|u\rangle_\ell = \|u_1 - \ell u_0\|^2 + \|h_0^{1/2}u_0\|^2 - \|(k-\ell)u_0\|^2 \tag{2.17}$$

and k is bounded, this is obvious. □

Lemma 2.11. *For all* $\ell \in \mathbb{R}$, $\langle\cdot|\cdot\rangle_\ell$ *is continuous with respect to the norm* $\|\cdot\|_{\dot{\mathcal{E}}}$ *if and only if*

$$h_0 \gtrsim (k-\ell)^2. \tag{2.18}$$

Proof. We have $\|u\|^2_{\dot{\mathcal{E}}} = \|u_1 - ku_0\|^2 + \|h_0^{1/2}u_0\|^2$ and we have to decide when $|\langle u|u\rangle_\ell| \lesssim \|u\|^2_{\dot{\mathcal{E}}}$. By (2.17) this holds if and only if $\left|\|u_1 - \ell u_0\|^2 - \|(k-\ell)u_0\|^2\right| \lesssim \|u_1 - ku_0\|^2 + \|h_0^{1/2}u_0\|^2$. If this holds and we take $u_1 = ku_0$ we get (2.18). The converse is obvious. □

2.5. **Energy Klein-Gordon operators.** Let

$$\hat{H} = \begin{pmatrix} 0 & 1 \\ h & 2k \end{pmatrix} = \Phi(k)\hat{K}\Phi^{-1}(k) \quad \text{where} \quad \hat{K} = \begin{pmatrix} k & 1 \\ h_0 & k \end{pmatrix}. \tag{2.19}$$

The *energy Klein-Gordon operators* will be various realizations of $\hat{H}$. The operator $\hat{K}$ is a *charge Klein-Gordon operator* and will only play a technical role.

2.5.1. *Klein-Gordon operator on the inhomogeneous energy space.* The *inhomogeneous Klein-Gordon operator* is the operator H induced by $\hat{H}$ on $\mathcal{E}$. This means that its domain is

$$D(H) := \{u \in \mathcal{E};\, \hat{H}u \in \mathcal{E}\} = \mathcal{H}^1 \oplus \mathcal{H}^{1/2},$$

and for $u \in D(H)$ we have $Hu = \hat{H}u$. For the second equality above, see [16, Sect. 5.2]. We also recall [16, Proposition 5.3]:

Proposition 2.12. *Assume* (A1) *and* (A2).

- *One has* $\rho(H) = \rho(h,k)$.
- *In particular, if* $\rho(h,k) \neq \emptyset$ *then* H *is a closed densely defined operator in* $\mathcal{E}$ *and its spectrum is invariant under complex conjugation.*

– *If* $z \in \rho(h,k)$, *then*

$$R(z) := (H-z)^{-1} = p^{-1}(z)\begin{pmatrix} z-2k & 1 \\ h & z \end{pmatrix}. \tag{2.20}$$

We may similarly define the operator K induced by $\hat{K}$ in $\mathcal{E}$ and one may easily check, under the same conditions (A1) and (A2), that $\Phi(k) : \mathcal{H}^1 \oplus \mathcal{H}^{1/2} \tilde{\to} \mathcal{H}^1 \oplus \mathcal{H}^{1/2}$ with inverse $\Phi(-k)$, hence H and K have the same domain and $H = \Phi(k)K\Phi(-k)$. This implies

$$(K-z)^{-1} = \begin{pmatrix} p^{-1}(z)(z-k) & p^{-1}(z) \\ 1+(z-k)p^{-1}(z)(z-k) & (z-k)p^{-1}(z) \end{pmatrix}. \tag{2.21}$$

2.5.2. *Klein-Gordon operator on the homogeneous energy space.* The *homogeneous Klein-Gordon operator* is the operator $\dot{H}$ induced by $\hat{H}$ on $\dot{\mathcal{E}}$. This means that its domain is

$$D(\dot{H}) = \{u \in \dot{\mathcal{E}};\ \hat{H}u \in \dot{\mathcal{E}}\}$$

and for $u \in D(\dot{H})$ we have $\dot{H}u = \hat{H}u$. The proofs will involve the homogeneous operator $\dot{K}$ associated to the auxiliary operator $\hat{K}$ and acting in the space $h_0^{-1/2}\mathcal{H} \oplus \mathcal{H}$ with domain

$$D(\dot{K}) = \{v \in h_0^{-1/2}\mathcal{H} \oplus \mathcal{H};\ \hat{K}v \in h_0^{-1/2}\mathcal{H} \oplus \mathcal{H}\}.$$

From the relations (2.15) and (2.19) we see that $\Phi(k)$ induces an isomorphism of $h_0^{-1/2}\mathcal{H} \oplus \mathcal{H}$ with $\dot{\mathcal{E}}$ whose inverse is $\Phi(-k)$. Clearly then $\dot{H} = \Phi(k)\dot{K}\Phi(-k)$.

Lemma 2.13. *Under the conditions* (A1) *and* (A2) *we have:*

$$D(\dot{H}) = \Phi(k)\left(\left(h_0^{-1/2}\mathcal{H} \cap h_0^{-1}\mathcal{H}\right) \oplus \mathcal{H}^{1/2}\right).$$

Proof. From the preceding comments we see that the assertion of the lemma is equivalent to

$$D(\dot{K}) = \left(h_0^{-1/2}\mathcal{H} \cap h_0^{-1}\mathcal{H}\right) \oplus \mathcal{H}^{1/2} \tag{2.22}$$

Since $\hat{K}v = \left(\begin{smallmatrix} kv_0+v_1 \\ h_0v_0+kv_1 \end{smallmatrix}\right)$, if v belongs to the right hand side above then $kv_0 + v_1 \in h_0^{-1/2}\mathcal{H}$ and $h_0v_0 + kv_1 \in \mathcal{H}$, thus $\hat{K}v \in h_0^{-1/2}\mathcal{H} \oplus \mathcal{H}$, hence $v \in D(\dot{K})$. Reciprocally, if $v \in D(\dot{K})$ then

$$v_0 \in h_0^{-1/2}\mathcal{H},\ v_1 \in \mathcal{H},\ kv_0 + v_1 \in h_0^{-1/2}\mathcal{H},\ h_0v_0 + kv_1 \in \mathcal{H}.$$

We have to show $v_0 \in h_0^{-1/2}\mathcal{H} \cap h_0^{-1}\mathcal{H}$ and $v_1 \in \mathcal{H}^{1/2}$, which follow from $v_0 \in h_0^{-1}\mathcal{H}$ and $v_1 \in h_0^{-1/2}\mathcal{H}$. The last relation is a consequence of $kv_0+v_1 \in h_0^{-1/2}\mathcal{H}$ because $k \in \mathcal{B}(h_0^{-1/2}\mathcal{H})$. Since k is bounded on $\mathcal{H}$ we finally get $h_0v_0 \in \mathcal{H} - kv_1 \subset \mathcal{H}$ hence $v_0 \in h_0^{-1}\mathcal{H}$. □

Lemma 2.14. *Assume that the conditions* (A1) *and* (A2) *are satisfied and let* $z \in \rho(h,k)\backslash\mathbb{R}$. *Then the maps* $p(z)^{-1}$ *and* $p(z)^{-1}h_0$ *induce continuous operators* $h_0^{-1/2}\mathcal{H} \to h_0^{-1/2}\mathcal{H} \cap h_0^{-1}\mathcal{H}$ *and* $h_0^{-1/2}\mathcal{H} \to \mathcal{H}^{1/2}$ *respectively.*

Proof. We set $m = z - k$ and, to simplify the writing, we do not specify z unless this is really necessary, e.g. we write p for $p(z)$ and $p = h_0 - m^2$. From (A2) it follows that m induces bounded invertible operators in all the spaces $\mathcal{H}^s$ with $-1/2 \le s \le 1/2$ and in the space $h_0^{-1/2}\mathcal{H}$ (in all $h_0^s\mathcal{H}$ with $-1/2 \le s \le 1/2$, in fact). Since h_0 extends to a unitary operator

$h_0^{-1/2}\mathcal{H} \to h_0^{1/2}\mathcal{H}$ and $h_0^{1/2}\mathcal{H}$ is a dense subspace of $\mathcal{H}^{-1/2}$, the operator $p^{-1}h_0$ extends to a bounded map $p^{-1}h_0 : h_0^{-1/2}\mathcal{H} \to \mathcal{H}^{1/2}$. Then we write

$$p^{-1} = p^{-1}(m^2 - h_0 + h_0)m^{-2} = p^{-1}h_0 m^{-2} - m^{-2}$$

from which it follows that p^{-1} extends to an operator in $\mathcal{B}(h_0^{-1/2}\mathcal{H})$. We still have to prove that p^{-1} sends $h_0^{-1/2}\mathcal{H}$ into $h_0^{-1}\mathcal{H}$. For this we note that

$$h_0 p^{-1} = (h_0 - m^2 + m^2)p^{-1} = 1 + m^2 p^{-1}$$

and thus, by what we just proved, we see that $h_0 p^{-1} h_0^{-1/2}\mathcal{H} \subset h_0^{-1/2}\mathcal{H}$ hence p^{-1} sends $h_0^{-1/2}\mathcal{H}$ into $h_0^{-3/2}\mathcal{H} \cap h_0^{-1/2}\mathcal{H} \subset h_0^{-1}\mathcal{H}$, which clearly proves the assertion. $\square$

Proposition 2.15. *If* (A1), (A2) *are true then* $\rho(\dot{H}) \setminus \mathbb{R} = \rho(h,k) \setminus \mathbb{R}$ *and for* z *in this set*

$$\dot{R}(z) := (\dot{H} - z)^{-1} = \Phi(k) \begin{pmatrix} p^{-1}(z)(z-k) & p^{-1}(z) \\ 1 + (z-k)p^{-1}(z)(z-k) & (z-k)p^{-1}(z) \end{pmatrix} \Phi(-k). \tag{2.23}$$

Proof. As in the proof of Lemma 2.13 we prove the corresponding statement for the operator $\dot{K}$. Fix $z \in \rho(h,k) \setminus \mathbb{R}$ and adopt the notations of the proof of Lemma 2.14. We show that $z \in \rho(\dot{K})$ and that $(\dot{K} - z)^{-1}$ is just the matrix in (2.23) or in (2.21):

$$(\dot{K} - z)^{-1} = \begin{pmatrix} p^{-1}m & p^{-1} \\ 1 + mp^{-1}m & mp^{-1} \end{pmatrix}. \tag{2.24}$$

We denote S the matrix in the right hand side of (2.24) and first show that S sends $h_0^{-1/2}\mathcal{H}\oplus\mathcal{H}$ into $D(\dot{K})$ as defined in (2.22). Thus, if $v_0 \in h_0^{-1/2}\mathcal{H}$ and $v_1 \in \mathcal{H}$ we must prove that

$$p^{-1}mv_0 + p^{-1}v_1 \in h_0^{-1/2}\mathcal{H} \cap h_0^{-1}\mathcal{H} \quad \text{and} \quad (1 + mp^{-1}m)v_0 + mp^{-1}v_1 \in \mathcal{H}^{1/2}. \tag{2.25}$$

From Lemma 2.1 we get $p^{-1}v_1 \in \mathcal{H}^1 \subset h_0^{-1/2}\mathcal{H} \cap h_0^{-1}\mathcal{H}$ hence also $mp^{-1}v_1 \in \mathcal{H}^{1/2}$. Thus it remains to treat the terms involving v_0. From Lemma 2.14 we get $p^{-1}mv_0 \in h_0^{-1/2}\mathcal{H} \cap h_0^{-1}\mathcal{H}$. On the other hand, since $p = h_0 - m^2$, Lemma 2.1 and a simple computation give

$$1 + mp^{-1}m = m^{-1}p^{-1}h_0 m^{-1} \tag{2.26}$$

in the sense of bounded operators $\mathcal{H}^{-1/2} \to \mathcal{H}^{1/2}$. From this relation and Lemma 2.14 we get $(1 + mp^{-1}m)v_0 \in \mathcal{H}^{1/2}$.

Thus $S : h_0^{-1/2}\mathcal{H} \oplus \mathcal{H} \to D(\dot{K})$ and a straightforward computation gives $(\dot{K} - z)Sv = v$ for all $v \in h_0^{-1/2}\mathcal{H} \oplus \mathcal{H}$. On the other hand, if $u \in D(\dot{K})$ and $v = (\dot{K} - z)u$ then it is easy to show that $u = Sv$. This finishes the proof of the relation $S = (\dot{K} - z)^{-1}$, i.e. of (2.24).

It remains to be shown that $\rho(\dot{K}) \setminus \mathbb{R} \subset \rho(h,k)$. Assume that $z \notin \mathbb{R}$ and

$$\dot{K} - z : \left(h_0^{-1/2}\mathcal{H} \cap h_0^{-1}\mathcal{H}\right) \oplus \mathcal{H}^{1/2} \to h_0^{-1/2}\mathcal{H} \oplus \mathcal{H}$$

is bijective. Then for any v of the form $v = \left(\begin{smallmatrix} 0 \\ v_1 \end{smallmatrix}\right) \in h_0^{-1/2}\mathcal{H}\oplus\mathcal{H}$ there is a unique $u = \left(\begin{smallmatrix} u_0 \\ u_1 \end{smallmatrix}\right)$ with $u_0 \in h_0^{-1/2}\mathcal{H} \cap h_0^{-1}\mathcal{H}$ and $u_1 \in \mathcal{H}^{1/2}$ such that $-mu_0 + u_1 = 0$ and $h_0 u_0 - mu_1 = v_1$. Then $u_0 = m^{-1}u_1 \in \mathcal{H}^{1/2}$ and also $u_0 \in h_0^{-1}\mathcal{H}$ hence $u \in \mathcal{H}^1$ and we have $pu_0 = (h_0 - m^2)u_0 = v_1$. Thus $p : \mathcal{H}^1 \to \mathcal{H}$ is surjective. It is also injective because if $pu_0 = 0$ for some $u_0 \in \mathcal{H}^1$ then $u = \left(\begin{smallmatrix} u_0 \\ mu_0 \end{smallmatrix}\right) \in D(\dot{K})$ and $(\dot{K} - z)u = 0$ hence $u = 0$. $\square$

We point out a simple relation between H and $\dot{H}$ (a similar statement holds for K and $\dot{K}$). Recall that $\mathcal{E} \subset \dot{\mathcal{E}}$ continuously and densely.

Lemma 2.16. *$\dot{H}$ coincides with the closure of H in $\dot{\mathcal{E}}$.*

Proof. If $z \in \rho(h,k) \setminus \mathbb{R}$ then z belongs to $\rho(H) \cap \rho(\dot{H})$ and the resolvents $R = (H-z)^{-1}$ and $\dot{R} = (\dot{H}-z)^{-1}$ are bounded operators in $\mathcal{E}$ and $\dot{\mathcal{E}}$ respectively. Moreover, $\dot{R}$ is clearly a continuous extension of R to $\dot{\mathcal{E}}$, so it is the closure of R in $\dot{\mathcal{E}}$. By thinking in terms of graphs one easily sees that $\dot{H} - z = \dot{R}^{-1}$ is the closure of $H - z = R^{-1}$ in $\dot{\mathcal{E}}$. □

We will often consider the case where (A3) is fulfilled. In this case we have that

$$D(\dot{H}) = \left(h^{-1/2}\mathcal{H} \cap h^{-1}\mathcal{H}\right) \oplus \mathcal{H}^{1/2}$$

and $\dot{H}$ is selfadjoint (see e.g. [19, Lemma 2.1.1]). Note also that in this case the resolvent of $\dot{H}$ is given by: (see [16, Proposition 5.7]

$$\dot{R}(z) = \begin{pmatrix} z^{-1}p^{-1}(z)h - z^{-1} & p^{-1}(z) \\ p^{-1}(z)h & zp^{-1}(z) \end{pmatrix}. \tag{2.27}$$

Moreover, if we assume (A3), then $\|\dot{R}(z)\|_{\mathcal{B}(\dot{\mathcal{E}})} \leq |\mathrm{Im} z|^{-1}$. Using [16, Proposition 5.10] we obtain the following resolvent estimate for H:

Proposition 2.17. *Assume* (A1)-(A2). *Then:*

$$\|R(z)\|_{\mathcal{B}(\mathcal{E})} \lesssim (1+|z|^{-1})\|\dot{R}(z)\|_{\mathcal{B}(\dot{\mathcal{E}})} + |z|^{-1}. \tag{2.28}$$

Assume in addition (A3). *Then:*

$$\|R(z)\|_{\mathcal{B}(\mathcal{E})} \lesssim (1+|z|^{-1})|\mathrm{Im} z|^{-1}. \tag{2.29}$$

2.5.3. *Gauge transformations.* Let us recall that our starting point was the Klein-Gordon equation

$$(\partial_t - ik)^2 u + h_0 u = 0. \tag{2.30}$$

If u is solution of (2.30) and $\ell \in \mathbb{R}$, then $v = e^{-it\ell}u$ solves :

$$(\partial_t - i(k-\ell))^2 v + h_0 v = 0. \tag{2.31}$$

Let us formulate this in terms of generators: if

$$\Phi(\ell) H \Phi^{-1}(\ell) =: H_\ell + \ell,$$

then

$$H_\ell = \begin{pmatrix} 0 & 1 \\ p(\ell) & 2(k-\ell) \end{pmatrix}, \; p(\ell) = h_0 - (k-\ell)^2.$$

It follows that if there exists $\ell \in \mathbb{R}$ such that (A3) is fulfilled with h replaced by $p(\ell)$ and k by $k - \ell$, then $\dot{H}$ is selfadjoint on the homogeneous energy space

$$\dot{\mathcal{E}} = \Phi(\ell)(p(\ell)^{-1/2}\mathcal{H} \oplus \mathcal{H}).$$

2.6. **Existence of the dynamics.** From [15, Corollary 8.6] we obtain :

Lemma 2.18. *H is the generator of a C_0-group e^{-itH} on $\mathcal{E}$.*

Now we show that e^{-itH} extends to a C_0-group on $\dot{\mathcal{E}}$.

Lemma 2.19. *$\dot{H}$ is the generator of a C_0- group on $\dot{\mathcal{E}}$ and for each real t the operator $e^{-it\dot{H}}$ coincides with the continuous extension of e^{-itH} to $\dot{\mathcal{E}}$.*

Proof. We start by proving that for some constants $C, \omega > 0$ we have

$$\|e^{-itH}\varphi\|_{\dot{\mathcal{E}}} \leq Ce^{\omega|t|}\|\varphi\|_{\dot{\mathcal{E}}} \quad \forall \varphi \in \mathcal{E}. \tag{2.32}$$

Let first $\varphi \in D(H)$. We compute by using (2.14) for $u = (u_0, u_1) = e^{-itH}\varphi$

$$\begin{aligned} \frac{d}{dt}\|u\|^2_{\dot{\mathcal{E}}} &= 2\mathrm{Re}(ihu_0 + iku_1|u_1 - ku_0) + 2\mathrm{Re}(h_0u_0|iu_1) \\ &= ([ik, h]u_0|u_0) \lesssim (h_0u_0|u_0) \lesssim \|u\|^2_{\dot{\mathcal{E}}}. \end{aligned}$$

The inequality (2.32) then follows for $\varphi \in D(H)$ by the Gronwall's lemma and for $\varphi \in \mathcal{E}$ by density. From (2.32) we see that e^{-itH} extends to a continuous operator V_t on $\dot{\mathcal{E}}$ such that $\|V_t\|_{\dot{\mathcal{E}}} \leq Ce^{\omega|t|}$. This clearly implies that V_t is a C_0-group on $\dot{\mathcal{E}}$ and from Nelson's invariant domain theorem it follows that its generator is the closure of H in $\dot{\mathcal{E}}$, which by Lemma 2.16 is just $\dot{H}$. □

3. Meromorphic extensions

In this Section we discuss various facts related to meromorphic extensions of quadratic pencils.

3.1. Background and definitions.

Definition 3.1. *Let $\mathcal{H}$ be a Hilbert space. For $z_0 \in \mathbb{C}$, let $\mathcal{U}$ be a neighborhood of z_0, and let $F : \mathcal{U} \setminus \{z_0\} \to B(\mathcal{H})$ be a holomorphic function. We say that F is* finite meromorphic *at z_0 if the Laurent expansion of F at z_0 has the form*

$$F(z) = \sum_{n=m}^{+\infty} (z - z_0)^n A_n, \quad m > -\infty,$$

the operators $A_m, ..., A_{-1}$ being of finite rank, if $m < 0$. If, in addition, A_0 is a Fredholm operator, then F is called Fredholm at z_0.

We will need the following fact, cf. [17, Proposition 4.1.4]:

Proposition 3.2. *Let $\mathcal{D} \subset \mathbb{C}$ be a connected open set, let $Z \subset \mathcal{D}$ be a discrete and closed subset of $\mathcal{D}$, and let $F : \mathcal{D} \setminus Z \to B(\mathcal{H})$ be a holomorphic function. Assume that*

- *F is finite meromorphic and Fredholm at each point of $\mathcal{D}$;*
- *there exists $z_0 \in \mathcal{D} \setminus Z$ such that $F(z_0)$ is invertible.*

Then there exists a discrete closed subset Z' of $\mathcal{D}$ such that $Z \subset Z'$ and:

- *$F(z)$ is invertible for each $z \in \mathcal{D} \setminus Z'$;*
- *$F^{-1} : \mathcal{D} \setminus Z' \to \mathcal{L}(\mathcal{H})$ is finite meromorphic and Fredholm at each point of $\mathcal{D}$.*

3.2. **Meromorphic extensions of weighted resolvents.** Let w be a positive selfadjoint operator on $\mathcal{H}$ with bounded inverse w^{-1}. One should think of w as a *weight function.* w and w^{-1} will act on $\mathcal{E}$, $\dot{\mathcal{E}}$ by $w(u_0, u_1) = (wu_0, wu_1)$ etc. In this subsection we will require (A3).

We need the following hypotheses:

$$\text{(ME1)} \quad \left\{ \begin{array}{ll} a) & wkw \in \mathcal{B}(\mathcal{H}). \\ b) & [k, w] = 0 \\ c) & h^{-1/2}[h, w^{-\epsilon}]w^{\epsilon/2} \in \mathcal{B}(\mathcal{H}) \, \forall \, 0 < \epsilon \leq 1 \\ d) & \text{if } \epsilon > 0 \text{ then } \|w^{-\epsilon}u\| \lesssim \|h^{1/2}u\| \quad \forall u \in h^{-1/2}\mathcal{H} \\ e) & w^{-1}\langle h\rangle^{-1} \in \mathcal{B}_\infty(\mathcal{H}). \end{array} \right.$$

Note that part d) of (ME1) is a Hardy type inequality and it implies the boundedness of the operators $w^{-\epsilon}h^{-1/2}$ and $h^{-1/2}w^{-\epsilon}$. Later on we shall see that these two operators are compact if (ME1) is satisfied (see the proof of Lemma 3.3).

Observe that from part c) we also get $w^{\epsilon/2}[h, w^{-\epsilon}]h^{-1/2} \in \mathcal{B}(\mathcal{H})$. Moreover, we shall have $w^{-\epsilon}\langle h\rangle^{-\tilde{\epsilon}} \in \mathcal{B}_\infty(\mathcal{H})$ for all $\epsilon, \tilde{\epsilon} > 0$. Indeed, $w^{-z}\langle h\rangle^{-z} \in \mathcal{B}_\infty(\mathcal{H})$ is an analytic function of z in the region $\mathrm{Re}\, z > 0$ and by e) above this is a compact operator for $\mathrm{Re}\, z \geq 1$, hence for any z.

We also need the assumption

$$\text{(ME2)} \quad \left\{ \begin{array}{l} \forall \, \epsilon > 0 \, \exists \, \delta_\epsilon > 0 \text{ such that } w^{-\epsilon}(h - z^2)^{-1}w^{-\epsilon} \text{ extends from } \mathrm{Im} z > 0 \\ \text{to } \mathrm{Im z} > -\delta_\epsilon \text{ as a finite meromorphic function with values in } \mathcal{B}_\infty(\mathcal{H}). \end{array} \right.$$

Lemma 3.3. *Assume* (A1)-(A3), (ME1)-(ME2) *and let* $0 < \epsilon \leq 1$. *Then the operators*

$$\begin{array}{l} (i) \quad w^{-\epsilon}p^{-1}(z)w^{-\epsilon}, \\ (ii) \ \ (h+1)^{1/2}w^{-\epsilon}p^{-1}(z)w^{-\epsilon}, \\ (iii) \ w^{-\epsilon}p^{-1}(z)hw^{-\epsilon}h^{-1/2}, \\ (iv) \ h^{1/2}w^{-\epsilon}p^{-1}(z)(z - 2k)w^{-\epsilon}h^{-1/2}, \end{array}$$

extend from $\{\mathrm{Im}\, z > 0$ *to* $\mathrm{Im}\, z > -\delta_{\epsilon/2}\}$ *as finite meromorphic functions with values in* $\mathcal{B}_\infty(\mathcal{H})$.

Proof. The resolvent identity yields

$$w^{-\epsilon}p^{-1}(z)w^{-\epsilon} = w^{-\epsilon}(h - z^2)^{-1}w^{-\epsilon}(1 + 2zw^\epsilon kw^\epsilon \cdot w^{-\epsilon}(h - z^2)^{-1}w^{-\epsilon})^{-1}.$$

Applying Proposition 3.2 to

$$F(z) = 1 + 2zw^\epsilon kw^\epsilon \cdot w^{-\epsilon}(h - z^2)^{-1}w^{-\epsilon}$$

proves (i). We now write

$$\begin{array}{rcl} hw^{-\epsilon}p^{-1}(z)w^{-\epsilon} & = & w^{-\epsilon}hp^{-1}(z)w^{-\epsilon} + [h, w^{-\epsilon}]w^{\epsilon/2}w^{-\epsilon/2}p^{-1}(z)w^{-\epsilon} \\ & = & w^{-2\epsilon} + z(z - 2k)w^{-\epsilon}p^{-1}(z)w^{-\epsilon} + [h, w^{-\epsilon}]w^{\epsilon/2}w^{-\epsilon/2}p^{-1}(z)w^{-\epsilon}. \end{array}$$

This allows us to compute the second operator

$$\begin{aligned} &(h+1)^{1/2}w^{-\epsilon}p^{-1}(z)w^{-\epsilon} \\ =&(h+1)^{-1/2}w^{-2\epsilon} + (h+1)^{-1/2}z(z - 2k) \cdot w^{-\epsilon}p^{-1}(z)w^{-\epsilon} \\ &+ (h+1)^{-1/2}[h, w^{-\epsilon}]w^{\epsilon/2} \cdot w^{-\epsilon/2}p^{-1}(z)w^{-\epsilon} + (h+1)^{-1/2}w^{-\epsilon}p^{-1}(z)w^{-\epsilon}. \end{aligned}$$

By using (i) and hypotheses $c), e)$ of (ME1) we get (ii).

Let us now prove (iii). Let $\chi \in C_0^\infty(\mathbb{R})$, $\chi = 1$ in a neighborhood of 0. We write

$$\begin{aligned} w^{-\epsilon}p^{-1}(z)hw^{-\epsilon}h^{-1/2} =& w^{-\epsilon}p^{-1}(z)hw^{-\epsilon}h^{-1/2}(1-\chi(h)) \\ &+ w^{-\epsilon}p^{-1}(z)hw^{-\epsilon}h^{-1/2}\chi(h) =: T_1 + T_2. \end{aligned}$$

We have

$$T_1 = w^{-2\epsilon}h^{-1/2}(1-\chi(h)) + w^{-\epsilon}p^{-1}(z)w^{-\epsilon}\cdot z(z-2k)h^{-1/2}(1-\chi(h)).$$

The first term is compact by a comment after hypothesis (ME1), the second is compact outside the poles of $w^{-\epsilon}p^{-1}(z)w^{-\epsilon}$ by part (i). We have

$$T_2 = w^{-\epsilon}p^{-1}(z)w^{-\epsilon/2}\cdot w^{\epsilon/2}[h, w^{-\epsilon}]h^{-1/2}\chi(h) + w^{-\epsilon}p^{-1}(z)w^{-\epsilon}\cdot h^{1/2}\chi(h).$$

By the same comment we see that both terms here extend to finite meromorphic functions with values in $B_\infty(\mathcal{H})$ in $\{\mathrm{Im}\, z > -\delta_{\epsilon/2}\}$. Thus (iii) is proved.

Note that since $h = p + z(z-2k)$ we have

$$w^{-\epsilon}p^{-1}(z)hw^{-\epsilon}h^{-1/2} = w^{-2\epsilon}h^{-1/2} + w^{-\epsilon}p^{-1}(z)w^{-\epsilon/2}\cdot z(z-2k)w^{-\epsilon/2}h^{-1/2}.$$

The left hand side here is a compact operator by what we just proved and the last term is also compact by (i) and because $w^{-\epsilon/2}h^{-1/2}$ is bounded by hypotheses $d)$ of (ME1). Since $0 < \epsilon \le 1$ is arbitrary, we see that $w^{-\epsilon}h^{-1/2}$ and $h^{-1/2}w^{-\epsilon}$ are compact operators if $0 < \epsilon \le 1$.

Finally, we prove (iv). We have

$$\begin{aligned} & h^{1/2}w^{-\epsilon}p^{-1}(z)(z-2k)w^{-\epsilon}h^{-1/2} \\ =& h^{-1/2}[h, w^{-\epsilon}]w^{\epsilon/2}\cdot w^{-\epsilon/2}p^{-1}(z)w^{-\epsilon/2}\cdot(z-2k)w^{-\epsilon/2}h^{-1/2} \\ &+ h^{-1/2}w^{-\epsilon/2}\cdot w^{-\epsilon/2}hp^{-1}(z)w^{-\epsilon/2}\cdot(z-2k)w^{-\epsilon/2}h^{-1/2}. \end{aligned}$$

For the first term of the right hand side we use $c)$ of (ME1) as well as (i) and the boundedness of $w^{-\epsilon/2}h^{-1/2}$. For the last term we note that it is equal to

$$\begin{aligned} & h^{-1/2}w^{-\epsilon}(z-2k)w^{-\epsilon}h^{-1/2} \\ +& h^{-1/2}w^{-\epsilon/2}z(z-2k)\cdot w^{-\epsilon/2}p^{-1}(z)w^{-\epsilon/2}\cdot(z-2k)w^{-\epsilon/2}h^{-1/2}. \end{aligned}$$

The first term is a holomorphic function with values in $B_\infty(\mathcal{H})$ because $w^{-\epsilon}h^{-1/2}$, $h^{-1/2}w^{-\epsilon} \in B_\infty(\mathcal{H})$. The last line is treated as before. This proves (iv). □

Using this lemma we obtain a meromorphic extension of the truncated resolvent of H.

Proposition 3.4. *Assume the hypotheses of Lemma 3.3 and let $\epsilon > 0$. Then $w^{-\epsilon}R(z)w^{-\epsilon}$ and $w^{-\epsilon}\dot{R}(z)w^{-\epsilon}$ extend finite meromorphically to $\{\mathrm{Im} z > -\delta_{\epsilon/2}\}$ as a operator valued functions with values in $\mathcal{B}_\infty(\mathcal{E})$ and $\mathcal{B}_\infty(\dot{\mathcal{E}})$ respectively.*

Proof. We first prove the assertion concerning $R(z)$. Using (2.20) we see that

$$\begin{aligned} w^{-\epsilon}R(z)w^{-\epsilon} &= w^{-\epsilon}p(z)^{-1}\begin{pmatrix} z-2k & 1 \\ h & z \end{pmatrix}w^{-\epsilon} \\ &= \begin{pmatrix} 0 & 0 \\ w^{-2\epsilon} & 0 \end{pmatrix} + w^{-\epsilon}p^{-1}w^{-\epsilon}\begin{pmatrix} z-2k & 1 \\ z(z-2k) & z \end{pmatrix}. \end{aligned}$$

We then use (i), (ii) of Lemma 3.3 as well as assumption (ME1)$e)$.

Let us now treat $\dot{R}(z)$. Recall that under hypothesis (A3) we have

$$\dot{R}(z) = \begin{pmatrix} z^{-1}p^{-1}(z)h - z^{-1} & p^{-1}(z) \\ p^{-1}(z)h & zp^{-1}(z) \end{pmatrix}.$$

Using that $w^{-\epsilon}h^{-1/2}$ is bounded by hypothesis (ME1)*d)* we can write

$$w^{-\epsilon}\dot{R}(z)w^{-\epsilon} = w^{-\epsilon}p^{-1}(z)\begin{pmatrix} z-2k & 1 \\ h & z \end{pmatrix}w^{-\epsilon}.$$

We therefore have to show that

$$h^{1/2}w^{-\epsilon}p^{-1}(z)(z-2k)w^{-\epsilon}h^{-1/2},\ h^{1/2}w^{-\epsilon}p^{-1}(z)w^{-\epsilon},$$
$$w^{-\epsilon}p^{-1}(z)hw^{-\epsilon}h^{-1/2},\ w^{-\epsilon}p^{-1}(z)zw^{-\epsilon}$$

extend finite meromorphically with values in $\mathcal{B}_\infty(\mathcal{H})$. This follows from Lemma 3.3. □

4. Klein-Gordon operators with "two ends"

In this section we discuss an abstract framework corresponding to Klein-Gordon operators on manifolds with "two ends". The essential condition is that the asymptotic Hamiltonians in both ends are selfadjoint for a positive energy norm, modulo some gauge transformation.

4.1. **Assumptions.**

We assume that there exists a selfadjoint operator x on $\mathcal{H}$ with $\sigma(x) = \sigma_{ac}(x) = \mathbb{R}$ such that w is a smooth function of x, such that k commutes with x and such that h_0 is *local* in x in the following sense:

if $\chi_1, \chi_2 \in C^\infty(\mathbb{R})$ are bounded together with all their derivatives and if $\operatorname{supp}\chi_1 \cap \operatorname{supp}\chi_2 = \emptyset$, then $\chi_1(x)h_0\chi_2(x) = 0$. To summarize we assume:

$$\text{(TE1)}\qquad \begin{cases} [x,k] = 0, \\ w = w(x) \quad \text{with } w \in C^\infty(\mathbb{R}), \\ h_0 \text{ is local in } x. \end{cases}$$

Let $\chi_\pm \in C^\infty(\mathbb{R})$ such that $\chi_+^2 + \chi_-^2 = 1$, $0 \le \chi_\pm \le 1$, and

$$\begin{array}{llll} \operatorname{supp}\chi_- & \subset & (-\infty,1), \quad \chi_- = 1 & \text{on} \quad (-\infty,0], \\ \operatorname{supp}\chi_+ & \subset & (-1,\infty), \quad \chi_+ = 1 & \text{on} \quad [0,\infty). \end{array}$$

If $\epsilon > 0$ is a real number, let

$$i_\pm = \chi_\pm(\epsilon^{-1}x), \quad j_\pm = \chi_\pm(\epsilon^{-1}x \mp 3).$$

We then have

$$j_\pm i_\pm = j_\pm, \quad i_+j_- = i_-j_+ = 0.$$

Let $\tilde{\chi} \in C_0^\infty((-2,2))$ with $\tilde{\chi} = 1$ on $[-1,1]$. We set $\tilde{i} = \tilde{\chi}(R^{-1}x)$. Let

$$k_\pm := k \mp \ell j_\mp^2,\ h_\pm := h_0 - k_\pm^2. \tag{4.1}$$

We also set:

$$\tilde{h}_- := h_- + 2\ell k_- - \ell^2 = h_0 - (\ell - k_-)^2. \tag{4.2}$$

We require

(TE2) there are $\ell \in \mathbb{R}$, $R > 0$ such that (h_+, k_+) and $(\tilde{h}_-, k_- - \ell)$ satisfy (A3).

We also set: $p_\pm(z) := h_\pm + z(2k_\pm - z)$. Note that $\tilde{h}_- = p_-(\ell)$.

4.2. **Asymptotic Hamiltonians.** We introduce the homogeneous energy spaces

$$\dot{\mathcal{E}}_+ := h_+^{-1/2}\mathcal{H} \oplus \mathcal{H},\ \dot{\mathcal{E}}_- := \Phi(\ell)(\tilde{h}_-^{-1/2}\mathcal{H} \oplus \mathcal{H}).$$

Then the operators

$$\dot{H}_\pm = \begin{pmatrix} 0 & 1 \\ h_\pm & 2k_\pm \end{pmatrix} \tag{4.3}$$

are selfadjoint with domains

$$\begin{aligned} D(\dot{H}_+) &= h_+^{-1/2}\mathcal{H} \cap h_+^{-1}\mathcal{H} \oplus \langle h_+ \rangle^{-1/2}\mathcal{H}, \\ D(\dot{H}_-) &= \Phi(\ell)((\tilde{h}_-^{-1/2}\mathcal{H} \cap \tilde{h}_-^{-1}\mathcal{H}) \oplus \langle \tilde{h}_- \rangle^{-1/2}\mathcal{H}). \end{aligned}$$

We denote $\dot{R}_\pm(z) := (\dot{H}_\pm - z)^{-1}$.

We will also need the following assumption for ℓ as in assumption (TE2):

$$\text{(TE3)} \quad \begin{cases} a) & wi_+ki_+w,\ wi_-(k-\ell)i_-w \in \mathcal{B}(\mathcal{H}), \\ b) & [h, i_\pm] = \tilde{i}[h, i_\pm]\tilde{i}, \\ c) & (h_+, k_+, w) \text{ and } (\tilde{h}_-, k_- - \ell, w) \text{ fulfill (ME1), (ME2)}, \\ d) & h_\pm^{1/2} i_\pm h_\pm^{-1/2},\ h_0^{1/2} i_\pm h_0^{-1/2} \in \mathcal{B}(\mathcal{H}), \\ e) & \text{the operators } w[h, i_\pm]wh_\pm^{-1/2},\ w[h, i_\pm]wh_0^{-1/2},\ [h, i_\pm]h_\pm^{-1/2}, \\ & [h, i_\pm]h_0^{-1/2},\ h_0^{-1/2}[w^{-1}, h_0]w \text{ are bounded on } \mathcal{H}, \\ f) & \text{if } \epsilon > 0 \text{ then } \|w^{-\epsilon}u\| \lesssim \|h_0^{1/2}u\| \quad \forall u \in h_0^{-1/2}\mathcal{H}. \end{cases}$$

As a direct consequence of Proposition 3.4 we obtain

Proposition 4.1. *For any $\epsilon > 0$ the functions $w^{-\epsilon}R_\pm(z)w^{-\epsilon}$ and $w^{-\epsilon}\dot{R}(z)w^{-\epsilon}$ extend finite meromorphically to $\{\mathrm{Im} z > -\delta_{\epsilon/2}\}$ with values in $\mathcal{B}_\infty(\mathcal{E}_\pm)$ and $\mathcal{B}(\dot{\mathcal{E}}_\pm)$ respectively.*

4.3. **Construction of the resolvent.** We will need the following lemma:

Lemma 4.2. *The linear maps*

$$i_\pm : \dot{\mathcal{E}} \to \dot{\mathcal{E}},\ i_\pm : \dot{\mathcal{E}}_\pm \to \dot{\mathcal{E}}_\pm,\ i_\pm : \dot{\mathcal{E}}_\pm \to \dot{\mathcal{E}},\ i_\pm : \dot{\mathcal{E}} \to \dot{\mathcal{E}}_\pm$$

are bounded..

Proof. First note that the condition (TE3)*d)* and the relation $[k, i_\pm] = 0$ give the continuity of the maps $i_\pm : \dot{\mathcal{E}} \to \dot{\mathcal{E}}$ and $i_\pm : \dot{\mathcal{E}}_\pm \to \dot{\mathcal{E}}_\pm$. Then note that

$$i_+(h_+ + k^2)i_+ = i_+h_0i_+, \quad i_-(\tilde{h}_- + (k-\ell)^2)i_- = i_-h_0i_-.$$

This together with (TE3)*d)* gives the continuity of $i_\pm : \dot{\mathcal{E}} \to \dot{\mathcal{E}}_\pm$. We then claim that

$$i_+(h_0 + k^2)i_+ \lesssim i_+h_+i_+, \tag{4.4}$$

$$i_-(h_0 + (k-\ell)^2)i_- \lesssim i_-\tilde{h}_-i_-. \tag{4.5}$$

Indeed, (4.4) follows from

$$i_+(h_0 + k^2)i_+ \lesssim i_+h_+i_+ + w^{-2}i_+^2 \lesssim i_+h_+i_+.$$

Here we have used (TE3)*c)*. Then (4.5) follows from

$$i_-(h_0 + (k-\ell)^2)i_- \lesssim i_-\tilde{h}_-i_- + i_-(k-\ell)^2i_- \lesssim i_-\tilde{h}_-i_- + w^{-2}i_-^2 \lesssim i_-\tilde{h}_-i_-.$$

Finally, (4.4), (4.5) ans (TE3)*d)* give the continuity of $i_+ : \dot{\mathcal{E}}_+ \to \dot{\mathcal{E}}$ and $i_- : \dot{\mathcal{E}}_- \to \dot{\mathcal{E}}$. $\square$

We introduce now a new operator:

$$Q(z) := i_-(\dot{H}_- - z)^{-1} i_- + i_+(\dot{H}_+ - z)^{-1} i_+. \tag{4.6}$$

Thanks to Lemma 4.2 $Q(z)$ is well defined as a bounded operator on $\dot{\mathcal{E}}$. We now compute :

$$(\dot{H} - z)Q(z) = 1 + [\dot{H}, i_-](\dot{H}_- - z)^{-1} i_- + [\dot{H}, i_+](\dot{H}_+ - z)^{-1} i_+.$$

Note that

$$[\dot{H}, i_\pm] = \begin{pmatrix} 0 & 0 \\ [h, i_\pm] & 0 \end{pmatrix}.$$

Let

$$K_\pm(z) := \begin{pmatrix} 0 & 0 \\ [h, i_\pm] & 0 \end{pmatrix} \dot{R}_\pm(z) i_\pm, \quad \tilde{K}_\pm(z) := i_\pm \dot{R}_\pm(z) \begin{pmatrix} 0 & 0 \\ [h, i_\pm] & 0 \end{pmatrix}. \tag{4.7}$$

Note that by assumptions (TE1), (TE3) the operators

$$[\dot{H}, i_\pm] w^\epsilon \quad \text{and} \quad i_\pm(1 - j_\pm) w^\epsilon \quad \text{are bounded on } \dot{\mathcal{E}} \text{ for all } \epsilon > 0. \tag{4.8}$$

We put

$$A(z) = K_-(z)(1 - j_-) + K_+(z)(1 - j_+) : \mathbb{C}^+ \to \mathcal{B}(\dot{\mathcal{E}})$$

Using Proposition 4.1 and Lemma 4.2 we see that $A(z)$ extends meromorphically to $\mathrm{Im} z > -\delta$ with values in $\mathcal{B}_\infty(\dot{\mathcal{E}})$ for some $\delta > 0$. As $\dot{H}_\pm$ are selfadjoint it follows that

$$\|A(z)\|_{\mathcal{B}(\dot{\mathcal{E}})} \le 1/2$$

for $\mathrm{Im} z$ sufficiently large. Thus $(1 + A(z))^{-1}$ exists for $\mathrm{Im} z$ large enough. By Proposition 3.2 there exists a closed discrete subset Z^+ of the half plane $\{\mathrm{Im} z > -\delta\}$ such that $(1 + A(z))^{-1}$ exists if $\mathrm{Im} z > -\delta$ and $z \notin Z^+$ and $(1 + A(z))^{-1}$ is finitely meromorphic in $\{\mathrm{Im} z > -\delta\}$ and analytic in $\{\mathrm{Im} > -\delta\} \setminus Z^+$. Let

$$K(z) = K_-(z) + K_+(z).$$

Now observe that we have $j_a K_b = 0$ for $a = \pm$, $b = \pm$ by assumption (TE3)*b)*. We therefore have

$$\begin{aligned} 1 + K(z) &= (1 + K_-(z) j_- + K_+(z) j_+)(1 + K_-(z)(1 - j_-) + K_+(z)(1 - j_+)), \\ (1 + K_-(z) j_- + K_+(z) j_+)^{-1} &= 1 - K_-(z) j_- - K_+(z) j_+. \end{aligned}$$

We can now construct the resolvent of $\dot{H}$ by

$$\begin{aligned} R_{\dot{H}}(z) &:= Q(z)(1 + K(z))^{-1} \\ &= Q(z)\big(1 + K_-(z)(1 - j_-) + K_+(z)(1 - j_+)\big)^{-1}\big(1 - K_-(z) j_- - K_+ j_+\big). \end{aligned} \tag{4.9}$$

The same considerations are valid in the lower half plane, we obtain a set of poles Z^-. The set $(Z^- \cap \mathbb{C}^-) \cup (Z^+ \cap \mathbb{C}^+)$ is clearly finite.

Proposition 4.3. *If the conditions* (A1)-(A2) *and* (TE1)-(TE3) *are satisfied then there is a finite set $Z \subset \mathbb{C} \setminus \mathbb{R}$ with $\overline{Z} = Z$ such that the spectra of H and $\dot{H}$ are included in $\mathbb{R} \cup Z$ and such that the resolvents R and $\dot{R}$ are finite meromorphic functions on $\mathbb{C} \setminus \mathbb{R}$. Moreover, the point spectrum of H coincides with the point spectrum of $\dot{H}$ and the set Z consists of eigenvalues of finite multiplicity of H and $\dot{H}$.*

Proof. From the previous arguments it follows that if we define $R_H(z) := R_{\dot{H}}(z)|_{\mathcal{E}}$ then $R_H(z) = R(z)$ and $R_{\dot{H}}(z) = \dot{R}(z)$ for z with sufficiently large (positive or negative) imaginary part. We know by Proposition 2.15 that $\rho(\dot{H}) \cap (\mathbb{C} \setminus \mathbb{R}) = \rho(h,k) \cap (\mathbb{C} \setminus \mathbb{R})$. Then we use Proposition 2.6 to see that all the poles of $\dot{H}$ in $\mathbb{C} \setminus \mathbb{R}$ are in a finite ball. But in this ball $(1 + A(z))^{-1}$ has only a finite number of poles. By using (4.9) and an analyticity argument we see that $\dot{R}(z)$ has only a finite number of poles in $\mathbb{C} \setminus \mathbb{R}$. From the analyticity properties of a resolvent family it follows then that the non real spectrum Z of $\dot{H}$ coincides with the set of non real poles of its resolvent, in particular is finite.

From $\rho(\dot{H})\backslash\mathbb{R} = \rho(h,k)\backslash\mathbb{R}$ and Lemma 2.1 we see that the complex spectrum is invariant under conjugation. Note also that every eigenvector of $\dot{H}$ for a non-zero eigenvalue is in $D(H)$ and thus an eigenvector of H. It remains to show that the complex point spectrum consists exactly of complex eigenvalues of $\dot{H}$ and that the corresponding eigenspaces are finite dimensional. If z_0 is a pole of $\dot{R}(z)$, then on a neighborhood of z_0 we may write $\dot{R}(z) = \sum_{n=1}^{N}(z_0 - z)^{-n}S_n + S(z)$ with S holomorphic near z_0 and the $S_n \neq 0$ of finite rank because the function $A(z)$ is finitely meromorphic. From this it follows that z_0 is an eigenvalue of finite multiplicity of $\dot{H}$, see [30, Ch. VIII Sec. 8] for details. □

From now on we shall denote $\sigma_{pp}^{\mathbb{C}}(\dot{H})$ the set of non real eigenvalues of $\dot{H}$. For $z \in \sigma_{pp}^{\mathbb{C}}(\dot{H})$ the Riesz projector is defined by

$$E(z, \dot{H}) = \frac{i}{2\pi} \oint_{\gamma} (\dot{H} - z)^{-1} dz,$$

where γ is a small curve in $\rho(\dot{H})$ surrounding z. Let

$$\mathbb{1}_{pp}^{\mathbb{C}}(\dot{H}) := \oplus_{z \in \sigma_{pp}^{\mathbb{C}}(H)} E(z, \dot{H}) \quad \text{and} \quad \mathcal{E}_{pp}^{\mathbb{C}}(\dot{H}) = \mathbb{1}_{pp}^{\mathbb{C}}(\dot{H})\dot{\mathcal{E}}.$$

Let furthermore

$$\mathbb{1}_{\mathbb{R}}(\dot{H}) = \mathbb{1} - \mathbb{1}_{pp}^{\mathbb{C}}(\dot{H}), \quad \mathcal{E}_{\mathbb{R}}(\dot{H}) = \mathbb{1}_{\mathbb{R}}(\dot{H})\dot{\mathcal{E}}.$$

We clearly have

$$\dot{\mathcal{E}} = \mathcal{E}_{\mathbb{R}}(\dot{H}) \oplus \mathcal{E}_{pp}^{\mathbb{C}}(\dot{H})$$

and both spaces are invariant under $e^{-it\dot{H}}$.

4.4. **Resolvent estimates.** For R, $\delta > 0$ we put

$$\mathcal{U}_0(R, \delta) \quad = \quad \{z \in \mathbb{C} : 0 < |\mathrm{Im} z| \leq \delta, \ |\mathrm{Re} z| \leq R\}.$$

We have

Lemma 4.4. *Assume* (A1), (A2), (TE1)-(TE3). *Then for each $R > 0$ there are $M, \delta > 0$ such that $\sigma(H) \setminus \mathbb{R}$ does not intersect $\mathcal{U}_0(R, \delta)$ and such that for all $z \in \mathcal{U}_0(R, \delta)$*

$$\|\dot{R}(z)\|_{\mathcal{B}(\dot{\mathcal{E}})} \lesssim |\mathrm{Im}\, z|^{-M}, \tag{4.10}$$

$$\|R(z)\|_{\mathcal{B}(\mathcal{E})} \lesssim \left(1 + |z|^{-1}\right) |\mathrm{Im}\, z|^{-M} + |z|^{-1}. \tag{4.11}$$

Proof. Recall that

$$\dot{R}(z) = Q(z)(1 + A(z))^{-1}(1 - K_-(z)j_- - K_+(z)j_+).$$

We choose $\delta > 0$ sufficiently small such that $(1 + A(z))^{-1}$ has no poles in $\mathcal{U}_0(R, \delta)$. We have:

$$\|Q(z)\|_{B(\dot{\mathcal{E}})} \lesssim |\mathrm{Im} z|^{-1}.$$

The meromorphic extension of $(1+A(z))^{-1}$ has only a finite number of real poles in $\overline{\mathcal{U}_0(R,\delta)}$, hence we have:

$$\|(1+A(z))^{-1}\|_{B(\dot{\mathcal{E}})} \lesssim |\mathrm{Im} z|^{-M_1}, \quad M_1 > 0.$$

Noting that $[H, i_\pm] : \dot{\mathcal{E}} \to \dot{\mathcal{E}}$ is bounded by assumption (TE3) we obtain :

$$\|(1 - K_-(z)j_- - K_+(z)j_+)\|_{B(\dot{\mathcal{E}})} \lesssim |\mathrm{Im} z|^{-1}.$$

This gives (4.10) with $M = M_1 + 2$. (4.11) now follows from Proposition 2.17. □

Remark 4.5. *If $\sigma^{\mathbb{C}}_{pp}(\dot{H}) = \emptyset$, then we can choose δ independently of R.*

Lemma 4.6. *Let $R \geq M\|k\|_{\mathcal{B}(\mathcal{H})}$ with M as in Proposition 2.6. Then we have*

$$\|\dot{R}(z)\|_{\mathcal{B}(\dot{\mathcal{E}})} \lesssim |\mathrm{Im}\, z|^{-1} \quad \textit{if } |z| \geq R.$$

Proof. Recall from (2.23) that

$$\dot{R}(z) \ := \ (\dot{H} - z)^{-1} = \Phi(k) \begin{pmatrix} -p^{-1}(z)(k-z) & p^{-1}(z) \\ 1 + (k-z)p^{-1}(z)(k-z) & -(k-z)p^{-1}(z) \end{pmatrix} \Phi(-k).$$

Therefore it is sufficient to show

$$\|h_0^{1/2} p^{-1}(z)(k-z)u\| \ \lesssim \ \frac{1}{|\mathrm{Im} z|}\|h_0^{1/2}u\|, \tag{4.12}$$

$$\|h_0^{1/2} p^{-1}(z)u\| \ \lesssim \ \frac{1}{|\mathrm{Im} z|}\|u\|, \tag{4.13}$$

$$\|(k-z)p^{-1}(z)u\| \ \lesssim \ \frac{1}{|\mathrm{Im} z|}\|u\|, \tag{4.14}$$

$$\|(1+(k-z)p^{-1}(z)(k-z))u\| \ \lesssim \ \frac{1}{|\mathrm{Im} z|}\|h_0^{1/2}u\|. \tag{4.15}$$

for $|z| \geq R$. (4.12)-(4.14) follow from Proposition 2.6. To show (4.15) we use (2.26) and write

$$1 + (k-z)p^{-1}(z)(k-z) = (k-z)p^{-1}(z)h_0^{1/2}h_0^{1/2}(k-z)^{-1}h_0^{-1/2}h_0^{1/2}.$$

Then using (2.7), (A2) we obtain

$$\|(1+(k-z)p^{-1}(z)(k-z))u\| \lesssim (\|k\|_{\mathcal{B}(\mathcal{H})} + |z|)\frac{1}{|\mathrm{Im} z|}\frac{1}{|z| - \|k\|_{\mathcal{B}(\mathcal{H})}}\|h_0^{1/2}u\|.$$

We can suppose $M \geq 2$ and obtain

$$(\|k\|_{\mathcal{B}(\mathcal{H})} + |z|)\frac{1}{|z| - \|k\|_{\mathcal{B}(\mathcal{H})}} \lesssim 1,$$

which finishes the proof of the lemma. □

4.5. **Smooth functional calculus.** The resolvent estimates in Lemma 4.4 easily allow to construct a smooth functional calculus for $\dot{H}$. For $f \in C_0^\infty(\mathbb{R})$ we denote by $\tilde{f} \in C_0^\infty(\mathbb{C})$ an almost analytic extension of f, satisfying

$$\begin{aligned} \tilde{f}|_{\mathbb{R}} &= f, \\ \left|\frac{\partial \tilde{f}(z)}{\partial \overline{z}}\right| &\leq C_N |\mathrm{Im} z|^N \quad N \in \mathbb{N}. \end{aligned}$$

Proposition 4.7. *Assume* (A1)-(A2), (TE1)-(TE3).

(i) Let $f \in C_0^\infty(\mathbb{R})$. Let $\tilde{f}$ be an almost analytic extension of f with $\operatorname{supp} \tilde{f} \cap \sigma_{pp}^{\mathbb{C}}(\dot{H}) = \emptyset$. *Then the integral*

$$f(\dot{H}) := \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \overline{z}}(z) \dot{R}(z) dz \wedge d\overline{z}$$

is norm convergent in $\mathcal{B}(\dot{\mathcal{E}})$ and is independent of the choice of $\tilde{f}$.
(ii) The map $C_0^\infty(\mathbb{R}) \ni f \mapsto f(\dot{H}) \in \mathcal{B}(\dot{\mathcal{E}})$ is an continuous algebra morphism if we equip $C_0^\infty(\mathbb{R})$ with its canonical topology.

Remark 4.8. *(i) The condition* $\operatorname{supp} \tilde{f} \cap \sigma_{pp}^{\mathbb{C}}(\dot{H}) = \emptyset$ *can always be satisfied by choosing* $\operatorname{supp} \tilde{f}$ *close enough to the real axis.*
(ii) If $\chi \in C^\infty(\mathbb{R})$ with $\chi = 1$ on $\mathbb{R} \backslash]-R, R[$ then we define $\chi(\dot{H}) := \mathbb{1}_{\mathbb{R}}(\dot{H}) - (1-\chi)(\dot{H})$.
(iii) We define in the same way a smooth functional calculus for $H, H_\pm, \dot{H}_\pm$. For $\dot{H}_\pm$ this coincides with the smooth functional calculus for selfadjoint operators.

Proposition 4.9. *If $\sigma_{pp}^{\mathbb{C}}(\dot{H}) = \emptyset$ and $\chi \in C_0^\infty(\mathbb{R})$, $\chi = 1$ in a neighborhood of zero, then*

$$\operatorname{s-}\lim_{L \to \infty} \chi\left(L^{-1} \dot{H}\right) = 1.$$

Proof. First note that we have for some $M > 0$ the estimate

$$\|\dot{R}(z)\|_{\mathcal{B}(\dot{\mathcal{E}})} \lesssim \frac{1}{|\mathrm{Im} z|} + \frac{1}{|\mathrm{Im} z|^M}, \quad |\mathrm{Im} z| > 0. \tag{4.16}$$

Indeed we first choose $R > 0$ as in Lemma 4.6. Then we have

$$\|\dot{R}(z)\|_{\mathcal{B}(\dot{\mathcal{E}})} \lesssim \frac{1}{|\mathrm{Im} z|}, \quad \forall z \in \mathbb{C} \setminus B(0, R).$$

By Remark 4.5 we can choose $\delta = R$ in Lemma 4.4 and obtain

$$\|\dot{R}(z)\|_{\mathcal{B}(\dot{\mathcal{E}})} \lesssim \frac{1}{|\mathrm{Im} z|^M}, \quad \forall z \in B(0, R) \subset \mathcal{U}_0(R, R).$$

In particular we can choose the same almost analytic extension of χ to define $\chi\left(L^{-1}\dot{H}\right)$ for all $L > 0$. We now show that

$$\lim_{L \to \infty} \chi\left(L^{-1}\dot{H}\right) - i_- \chi\left(\frac{\dot{H}_-}{L}\right) i_- - i_+ \chi\left(\frac{\dot{H}_+}{L}\right) i_+ = 0. \tag{4.17}$$

We have

$$\chi\left(L^{-1}\dot{H}\right) - i_- \chi\left(\frac{\dot{H}_-}{L}\right) i_- - i_+ \chi\left(\frac{\dot{H}_+}{L}\right) i_+ = \frac{1}{2\pi i} \int \overline{\partial} \tilde{\chi}(z) L(\dot{R}(Lz) - Q(Lz)) dz \wedge d\overline{z}.$$

Now recall that $\dot{R}(z) = Q(z)(1 + K(z))^{-1}$, thus

$$\dot{R}(Lz) - Q(Lz) = -\dot{R}(Lz) K(Lz).$$

Thanks to (4.16) we have the estimate for $L \geq 1$

$$\|\overline{\partial} \tilde{\chi}(z) L \dot{R}(Lz) K(Lz)\| \lesssim \frac{1}{L} \to 0.$$

This implies (4.17). As $\dot{H}_\pm$ is selfadjoint in $\dot{\mathcal{E}}_\pm$ we find using Lemma 4.2

$$\text{s-}\lim_{L\to\infty} i_\pm \chi\left(L^{-1}\dot{H}_\pm\right) i_\pm = i_\pm^2.$$

Thus

$$\text{s-}\lim_{L\to\infty} \chi\left(L^{-1}\dot{H}\right) = i_-^2 + i_+^2 = 1.$$

□

5. Propagation estimates

In this section we derive resolvent and propagation estimates for $\dot{H}$, similar to those obtained for selfadjoint operators. The key ingredients are the meromorphic extension of $\dot{R}(z)$ in Sect. 3 and the fact that the asymptotic Hamiltonians $\dot{H}_\pm$ are selfadjoint for their energy norms. There is however a new difficulty not present in the selfadjoint case: in addition to resolvent poles and thresholds, additional spectral singularities may appear. In the theory of selfadjoint operators on Krein spaces used in our previous works [15, 16] these spectral singularities are known as *critical points*.

5.1. **Resonances and boundary values of the resolvent.**

By the usual arguments the operator

$$A_w(z) = w^\epsilon K_-(z)(1-j_-)w^{-\epsilon} + w^\epsilon K_+(z)(1-j_+)w^{-\epsilon}.$$

can also be extended meromorphically from the upper half plane to $\{\mathrm{Im} z > -\delta_{\epsilon/2}\}$ with values in $\mathcal{B}_\infty(\dot{\mathcal{E}})$. By the same argument as in the construction of the resolvent we have for $\mathrm{Im} z$ large enough

$$\|A_w(z)\|_{\mathcal{B}(\dot{\mathcal{E}})} \le 1/2.$$

Using Proposition 3.2 we see that $(1 + A_w(z))^{-1}$ is meromorphic in $\{\mathrm{Im} z > -\delta_{\epsilon/2}\}$. Let S_w be the set of its poles. Now we have :

$$\begin{aligned} w^{-\epsilon}\dot{R}(z)w^{-\epsilon} &= w^{-\epsilon}Q(z)w^{-\epsilon}(1+A_w(z))^{-1} \\ &\times\ (1 - w^\epsilon K_-(z)j_-w^{-\epsilon} - w^\epsilon K_+(z)j_+w^{-\epsilon}). \end{aligned} \tag{5.1}$$

Using (5.1) we see that $w^{-\epsilon}\dot{R}(z)w^{-\epsilon}$ can be extended meromorphically from the upper half plane to $\{\mathrm{Im} z > -\delta_{\epsilon/2}\}$ with values in $\mathcal{B}_\infty(\dot{\mathcal{E}})$. The same result holds also for the resolvents of $\dot{H}_\pm$, by assumption ((TE3))*c)*.

Definition 5.1. *i) the poles in* $\{\mathrm{Im} z \le 0\}$ *of the meromorphic extension of* $w^{-\epsilon}\dot{R}(z)w^{-\epsilon}$ *are called* resonances *of* $\dot{H}$.
ii) the set of real resonances *of* $\dot{H}$, *resp.* $\dot{H}_\pm$ *is denoted by* $\mathcal{T}$, *resp.* $\mathcal{T}_\pm$.

Note that $\mathcal{T}$, $\mathcal{T}_\pm$ are obviously closed, discrete sets. As a consequence of the meromorphic extensions of $w^{-\epsilon}\dot{R}(z)w^{-\epsilon}$ and $w^{-\epsilon}\dot{R}_\pm(z)w^{-\epsilon}$ we obtain:

Proposition 5.2. *Assume* (A1)-(A2), (TE1)-(TE3). *Let* $\epsilon > 0$.

– *there exists* $\nu > 0$ *such that for all* $\chi \in C_0^\infty(\mathbb{R}\setminus\mathcal{T})$ *and all* $k \in \mathbb{N}$ *we have*

$$\sup_{\nu\ge\delta>0,\,\lambda\in\mathbb{R}} \|w^{-\epsilon}\chi(\lambda)\dot{R}^k(\lambda \pm i\delta)w^{-\epsilon}\|_{\mathcal{B}(\dot{\mathcal{E}})} < \infty. \tag{5.2}$$

– *for all* $\chi \in C_0^\infty(\mathbb{R} \setminus \mathcal{T}_\pm)$ *and all* $k \in \mathbb{N}$ *we have*

$$\sup_{\delta>0,\,\lambda\in\mathbb{R}} \|w^{-\epsilon}\chi(\lambda)\dot{R}_\pm^k(\lambda \pm i\delta)w^{-\epsilon}\|_{\mathcal{B}(\dot{\mathcal{E}}_\pm)} < \infty. \tag{5.3}$$

We apply [29, Thm. 4.3.1] to obtain:

Corollary 5.3. *Assume* (A1)-(A2), (TE1)-(TE3). *Let* $\epsilon > 0$ *and* $\mathcal{T}_\pm$ *be as in Proposition 5.2 ii). Then we have for all* $\chi \in C_0^\infty(\mathbb{R} \setminus \mathcal{T}_\pm)$

$$\sup_{\|u\|_{\dot{\mathcal{E}}_\pm}=1,\,\delta\neq 0} \int_{\mathbb{R}} (\|w^{-\epsilon}\dot{R}_\pm(\lambda + i\delta)\chi(\lambda)u\|^2_{\dot{\mathcal{E}}_\pm} d\lambda < \infty. \tag{5.4}$$

Note that we can not apply directly [29, Thm. 4.3.1] to $\dot{H}$, because the self-adjointness of the operator is crucial in this theorem. To discuss this further let is introduce a definition.

Definition 5.4. *We call* $\lambda \in \mathbb{R}$ *a* regular point *of* $\dot{H}$ *if there exists* $\chi \in C_0^\infty(\mathbb{R})$, $\chi(\lambda) = 1$ *and* $\nu > 0$ *such that:*

$$\sup_{\|u\|_{\dot{\mathcal{E}}_\pm}=1,\,\nu>|\delta|>0} \int_{\mathbb{R}} (\|w^{-\epsilon}\dot{R}(\lambda + i\delta)\chi(\lambda)u\|^2_{\dot{\mathcal{E}}_\pm} d\lambda < \infty. \tag{5.5}$$

Otherwise we call it a singular point. *We denote by* $\mathcal{S}$ *the set of singular points of* $\dot{H}$.

Remark 5.5. *Denoting by* $\mathcal{S}_\pm$ *the analog of* $\mathcal{S}$ *for* $\dot{H}_\pm$ *we see that* $\mathcal{S}_\pm = \mathcal{T}_\pm$ *by Kato's theory of* H*-smoothness (see* [29]*).*

In our situation it is still possible to control the set of singular points. Recall that

$$Q(z) = (1 - \tilde{K}_-(z) - \tilde{K}_+(z))\dot{R}(z).$$

We then have

$$w^{-\epsilon}Q(z) = (1 - w^{-\epsilon}\tilde{K}_-(z)w^\epsilon - w^{-\epsilon}\tilde{K}_+(z)w^\epsilon)w^{-\epsilon}\dot{R}(z).$$

Let

$$\tilde{A}_w(z) := -w^{-\epsilon}\tilde{K}_-(z)w^\epsilon - w^{-\epsilon}\tilde{K}_+(z)w^\epsilon.$$

By the usual arguments $\tilde{A}_w$ is meromorphic in $\{\mathrm{Im} z > -\delta_{\epsilon/2}\}$ with values in $\mathcal{B}_\infty(\dot{\mathcal{E}})$. Also $\|\tilde{A}_w(z)\|_{\mathcal{B}(\dot{\mathcal{E}})} \le 1/2$ for $\mathrm{Im} z$ sufficiently large. We can therefore apply again Proposition 3.2 to see that $(1 + \tilde{A}_w(z))^{-1}$ is meromorphic for $\{\mathrm{Im} z > -\delta_{\epsilon/2}\}$. We then have

$$w^{-\epsilon}\dot{R}(z) = (1 + \tilde{A}_w(z))^{-1}w^{-\epsilon}Q(z). \tag{5.6}$$

Proposition 5.6. *Assume* (A1)-(A2), (TE1)-(TE3). *Let* $\mathbb{N}_w$ *be the set of real poles* $\tilde{A}_w(z)$*. Then*

$$\mathcal{S} \subset \tilde{N}_w \cup \mathcal{T}_+ \cup \mathcal{T}_-.$$

It follows that $\mathcal{S}$ *is a closed and discrete set.*

Proof. The estimate (5.5) with $\chi(\lambda)$ instead of $\chi(\dot{H})$ follows from (5.6) and Corollary 5.3. We therefore only have to show that we can replace $\chi(\lambda)$ by $\chi(\dot{H})$. We choose $\tilde{\chi} \in C_0^\infty(I)$ with $\tilde{\chi}\chi = \chi$ and write :

$$\begin{aligned} \|w^{-\epsilon}\dot{R}(\lambda \pm i\delta)\chi(\dot{H})f\|^2_{\dot{\mathcal{E}}} &\lesssim \|w^{-\epsilon}\dot{R}(\lambda \pm i\delta)\tilde{\chi}(\lambda)\chi(\dot{H})f\|^2_{\dot{\mathcal{E}}} \\ &+ \|w^{-\epsilon}\dot{R}(\lambda \pm i\delta)(1 - \tilde{\chi}(\lambda))\chi(\dot{H})f\|^2_{\dot{\mathcal{E}}}. \end{aligned} \tag{5.7}$$

The estimate for the first term follows from the estimate with $\chi(\lambda)$. Let us treat the second term. We claim

$$\|w^{-\epsilon}\dot{R}(\lambda\pm \mathrm{i}\delta)(1-\tilde{\chi}(\lambda))\chi(\dot{H})\|_{\mathcal{B}(\dot{\mathcal{E}})}\lesssim \langle\lambda\rangle^{-1},$$

uniformly in δ. In fact let

$$f^{\epsilon}_{\lambda}(x)=\langle\lambda\rangle\frac{1}{x-(\lambda+\mathrm{i}\delta)}(1-\tilde{\chi}(\lambda))\chi(x).$$

It is sufficient to show that all the semi-norms $\|f^{\epsilon}_{\lambda}\|_m$ are uniformly bounded with respect to λ, δ. Note that $g_{\lambda}(x)=(1-\tilde{\chi}(\lambda))\chi(x)$ vanishes to all orders at $x=\lambda$. If $\operatorname{supp}\chi\subset[-C,C]$ this is enough to ensure that $\|f^{\epsilon}_{\lambda}\|_m$ is uniformly bounded in $\lambda\in[-2C,2C]$ and $\delta>0$. For $|\lambda|\geq 2C$ we observe that

$$\left|\langle\lambda\rangle\frac{1}{x-(\lambda+\mathrm{i}\delta)}\right|\lesssim 1$$

with analogous estimates for the derivatives. This gives the integrability of the second term in (5.7). $\square$

5.2. **Propagation estimates.** As an immediate consequence of Proposition 5.2 we obtain :

Proposition 5.7. *Assume* (A1)-(A2), (TE1)-(TE3). *Let* $\epsilon>0$.

– *for all* $\chi\in C_0^\infty(\mathbb{R}\setminus\mathcal{T})$ *and* $k\in\mathbb{N}$ *we have*

$$\|w^{-\epsilon}e^{-it\dot{H}}\chi(\dot{H})w^{-\epsilon}\|_{\mathcal{B}(\dot{\mathcal{E}})}\lesssim\langle t\rangle^{-k}. \tag{5.8}$$

– *for all* $\chi\in C_0^\infty(\mathbb{R}\setminus\mathcal{T}_\pm)$ *and* $k\in\mathbb{N}$ *we have*

$$\|w^{-\epsilon}e^{-it\dot{H}_\pm}\chi(\dot{H}_\pm)w^{-\epsilon}\|_{\mathcal{B}(\dot{\mathcal{E}}_\pm)}\lesssim\langle t\rangle^{-k}. \tag{5.9}$$

Proof. We only prove i), the proof of ii) being analogous. We have

$$w^{-\epsilon}e^{-it\dot{H}}\chi(\dot{H})w^{-\epsilon}=\frac{1}{2\pi i}\int\chi(\lambda)e^{-it\lambda}w^{-\epsilon}(\dot{R}(\lambda+i0)-\dot{R}(\lambda-i0))w^{-\epsilon}d\lambda.$$

Integration by parts gives :

$$w^{-\epsilon}e^{-it\dot{H}}\chi(\dot{H})w^{-\epsilon}=\frac{1}{2\pi i}\frac{1}{(it)^k}\sum_{\pm}\sum_{j=1}^{k+1}\pm C_k^{j-1}\int\chi_j(\lambda)e^{-it\lambda}w^{-\epsilon}\dot{R}^j(\lambda\pm i0)w^{-\epsilon}d\lambda$$

with $\chi_j=\chi^{(k+1-j)}$. The estimate then follows from Proposition 5.2. $\square$

Proposition 5.8. *Assume* (A1)-(A2), (TE1)-(TE3). *Let* $\epsilon>0$. *Then we have for all* $\chi\in C_0^\infty(\mathbb{R}\setminus\mathcal{S})$*:*

$$\int_{\mathbb{R}}\|w^{-\epsilon}e^{-it\dot{H}}\chi(\dot{H})\varphi\|^2_{\dot{\mathcal{E}}}dt\lesssim\|\varphi\|^2_{\dot{\mathcal{E}}}. \tag{5.10}$$

Proof. We write :

$$w^{-\epsilon}(\dot{R}(\lambda+\mathrm{i}\delta)-\dot{R}(\lambda-\mathrm{i}\delta))\chi(\dot{H})f=i\int_{\mathbb{R}}w^{-\epsilon}\mathrm{e}^{-\delta|t|}\mathrm{e}^{\mathrm{i}\lambda t}\mathrm{e}^{-\mathrm{i}\dot{H}t}\chi(\dot{H})f\,dt.$$

By Plancherel's formula this yields:

$$\int_{\mathbb{R}} \|w^{-\epsilon}(\dot{R}(\lambda+\mathrm{i}\delta)-\dot{R}(\lambda-\mathrm{i}\delta))\chi(\dot{H})f\|_{\dot{\mathcal{E}}}^2 d\lambda = \int_{\mathbb{R}} \mathrm{e}^{-2\delta|t|}\|w^{-\epsilon}\mathrm{e}^{-\mathrm{i}t\dot{H}}\chi(\dot{H})f\|_{\dot{\mathcal{E}}}^2 dt.$$

The lhs of this equation is uniformly bounded in δ with δ small enough. □

Corollary 5.9. *If* (A1)-(A2), (TE1)-(TE3) *hold and λ is a real eigenvalue of $\dot{H}$ then $\lambda\in\mathcal{S}$.*

5.3. **Estimates on singular points.** It will be important in applications to prove that $\dot{H}$ has no singular points. To do this we will use the following proposition.

Proposition 5.10. *Assume* (A1)-(A2), (TE1)-(TE3)*. Then*

$$\mathcal{S}\subset\mathcal{T}\cup\mathcal{T}_-\cup\mathcal{T}_+.$$

Proof. From (4.9) we obtain for $\mathrm{Im} z\gg 1$:

$$\dot{R}(z)=Q(z)(\mathbb{1}+K(z))^{-1}=Q(z)-Q(z)(\mathbb{1}+K(z))^{-1}K(z),$$

hence

$$\begin{aligned} w^{-\epsilon}\dot{R}(z)=&w^{-\epsilon}Q(z)-w^{-\epsilon}Q(z)(\mathbb{1}+K(z))^{-1}w^{-\epsilon}w^{\epsilon}K(z)\\ =&w^{-\epsilon}Q(z)-w^{-\epsilon}\dot{R}(z)w^{-\epsilon}w^{\epsilon}K(z). \end{aligned}$$

Next we write $w^{\epsilon}K(z)=w^{\epsilon}K_-(z)+w^{\epsilon}K_+(z)$ and obtain from the expression (4.7) of $K_\pm(z)$ that $w^{\epsilon}K_\pm(z)=m_\epsilon\dot{R}_\pm(z)i_\pm$ for $m_\epsilon\in B(\dot{\mathcal{E}})$. It suffices then to recall the expression (4.6) of $Q(z)$, and apply Prop. 5.7 and Remark 5.5. □

5.4. **Additional resolvent estimates.** In this subsection we make the link between the poles of $\eta p^{-1}(z)\eta$ and those of $\eta\dot{R}(z)\eta$ for $\eta\in C_0^\infty(\mathbb{R})$.

We will need the following hypothesis:

$$\text{(PE)}\qquad \left\{\begin{array}{ll} a) & \psi\in C_0^\infty(\mathbb{R})\Rightarrow h_0^{1/2}\psi(x)h_0^{-1/2}\in\mathcal{B}(\mathcal{H}),\\ b) & \psi\in C_0^\infty(\mathbb{R}),\ \psi\geq 0,\ \psi=1 \text{ near } 0\ \Rightarrow \text{s-lim}_{n\to\infty}\psi\left(\frac{x}{n}\right)=1 \text{ in } h_0^{-1/2}\mathcal{H}. \end{array}\right.$$

Lemma 5.11. *Let $\eta,\tilde{\eta}\in C_0^\infty(\mathbb{R})$ with $\tilde{\eta}\eta=\eta$. If z is not a pole of $\tilde{\eta}p^{-1}(z)\tilde{\eta}$ then z is not a pole of $\eta R(z)\eta$ nor of $\eta\dot{R}(z)\eta$ and if $\mathcal{P}(z):=\|\tilde{\eta}p^{-1}(z)\tilde{\eta}\|_{\mathcal{B}(\mathcal{H})}$ then we have the estimates*

$$\|\eta R(z)\eta\|_{\mathcal{B}(\mathcal{E})}\lesssim\langle z\rangle^2(1+\langle z\rangle\mathcal{P}^2(z)),\tag{5.11}$$

$$\|\eta\dot{R}(z)\eta\|_{\mathcal{B}(\dot{\mathcal{E}})}\lesssim\langle z\rangle^2(1+\langle z\rangle\mathcal{P}^2(z)).\tag{5.12}$$

Proof. We choose functions η_1, $\eta_0\in C_0^\infty(\mathbb{R})$ with $\eta_0\eta=\eta$, $\eta_1\eta_0=\eta_0$ and $\tilde{\eta}\eta_1=\eta_1$. We first notice that (5.12) follows from (5.11) because

$$\|\eta\dot{R}(z)\eta u\|_{\dot{\mathcal{E}}}\lesssim\|\eta R(z)\eta u\|_{\mathcal{E}}\lesssim\langle z\rangle^2(1+\langle z\rangle\mathcal{P}^2(z))\|\eta_0 u\|_{\mathcal{E}}\lesssim\langle z\rangle^2(1+\langle z\rangle\mathcal{P}^2(z))\|u\|_{\dot{\mathcal{E}}},$$

where we have used Hardy's inequality, (TE3)*f)*. Now recall from (2.23) that

$$\dot{R}(z):=(\dot{H}-z)^{-1}=\Phi(k)\begin{pmatrix}-p^{-1}(z)(k-z) & p^{-1}(z)\\ 1+(k-z)p^{-1}(z)(k-z) & -(k-z)p^{-1}(z)\end{pmatrix}\Phi(-k).$$

It is therefore sufficient to show:

$$\|\eta p^{-1}(z)(k-z)\eta u\|_{\mathcal{H}^1} \lesssim \langle z\rangle(1+\langle z\rangle^2\mathcal{P}(z))\|u\|_{\mathcal{H}^1}, \tag{5.13}$$

$$\|\eta p^{-1}(z)\eta u\|_{\mathcal{H}^1} \lesssim (1+\langle z\rangle^2\mathcal{P}(z))\|u\|_{\mathcal{H}}, \tag{5.14}$$

$$\|\eta(1+(k-z)p^{-1}(z)(k-z)\eta u\|_{\mathcal{H}} \lesssim \langle z\rangle(1+\langle z\rangle^2\mathcal{P}^2(z))\|u\|_{\mathcal{H}^1}, \tag{5.15}$$

$$\|\eta(k-z)p^{-1}(z)\eta u\|_{\mathcal{H}} \lesssim \langle z\rangle\mathcal{P}(z)\|u\|_{\mathcal{H}}. \tag{5.16}$$

(5.16) is clear, let us consider (5.14). By complex interpolation (5.14) will follow from

$$\|\eta p^{-1}(z)\eta u\|_{\mathcal{H}^2} \lesssim (\langle z\rangle^2\mathcal{P}(z)+1)\|u\|_{\mathcal{H}}. \tag{5.17}$$

We compute

$$\begin{aligned}h_0\eta p^{-1}(z)\eta =&[h_0,\eta]p^{-1}(z)\eta+\eta h_0p^{-1}(z)\eta\\ =&(h_0+1)^{-1}[h_0,\eta]\eta_0(h_0+1)p^{-1}(z)\eta\\ &+(h_0+1)^{-1}[h_0,[h_0,\eta]]\eta_0p^{-1}(z)\eta+\eta h_0p^{-1}(z)\eta,\\ \eta_0h_0p^{-1}(z)\eta =&\eta+(k-z)^2\eta_0p^{-1}(z)\eta.\end{aligned}$$

Thus we have

$$\|\eta_0h_0p^{-1}(z)\eta u\| \lesssim (1+\langle z\rangle^2\mathcal{P}(z))\|u\|_{\mathcal{H}}.$$

Using that $(h_0+1)^{-1}[h_0,[h_0,\eta]]$ is bounded this gives (5.17) and thus (5.14). Let us now consider (5.13). First note that $\|(k-z)u\|_{\mathcal{H}^1}\lesssim\langle z\rangle\|u\|_{\mathcal{H}^1}$. We then estimate using (5.14)

$$\|\eta p^{-1}(z)\eta u\|_{\mathcal{H}^1} \lesssim (\langle z\rangle^2\mathcal{P}(z)+1)\|u\|_{\mathcal{H}^1}.$$

This gives (5.13). Let us now show (5.15). We write

$$\begin{aligned}&\eta(1+(k-z)p^{-1}(z)(k-z))\eta\\ =&\eta p^{-1}(z)\eta_1[h_0,k\eta_0]h_0^{-1/2}h_0^{1/2}\eta_1p^{-1}(z)(k-z)\eta+\eta p^{-1}(z)(k-z)^2\eta.\end{aligned}$$

We have using (5.13) :

$$\begin{aligned}&\|\eta p^{-1}(z)\eta_1[h_0,k\eta_0]h_0^{-1/2}h_0^{1/2}\eta_1p^{-1}(z)(k-z)\eta u\|_{\mathcal{H}}\\ \lesssim&\mathcal{P}(z)\|h_0^{1/2}\eta_1p^{-1}(z)(k-z)\eta\|_{\mathcal{B}(\mathcal{H})}\|u\|_{\mathcal{H}}\\ \lesssim&\langle z\rangle\mathcal{P}(z)(1+\langle z\rangle^2\mathcal{P}(z))\|u\|_{\mathcal{H}^1}.\end{aligned}$$

This proves (5.15). □

Corollary 5.12. *If $w^{-\epsilon}p^{-1}(z)w^{-\epsilon}$ has no real poles then $w^{-\epsilon}\dot{R}(z)w^{-\epsilon}$ has no real poles.*

Proof. By the preceding lemmas $\eta\dot{R}(z)\eta$ has no real poles for all $\eta\in C_0^\infty(\mathbb{R})$. Suppose that $w^{-\epsilon}\dot{R}(z)w^{-\epsilon}$ has a pole at $z=z_0\in\mathbb{R}$. In a neighborhood of $z=z_0$ we have

$$w^{-\epsilon}\dot{R}(z)w^{-\epsilon}=\sum_{j=1}^m\frac{A_j}{(z-z_0)^j}+H(z),$$

where $H(z)$ is holomorphic and A_j are of finite rank. Let $\eta_1,\,\eta_2\in C_0^\infty(\mathbb{R})$. We have

$$w^{-\epsilon}\eta_1\dot{R}(z)\eta_2w^{-\epsilon}=\sum_{j=1}^m\frac{\eta_1A_j\eta_2}{(z-z_0)^j}+\eta_1H(z)\eta_2.$$

As $\eta_1 \dot{R}(z)\eta_2$ doesn't have a pole at $z = z_0$, we have

$$\eta_1 A_j \eta_2 = 0, \quad \forall \eta_1, \eta_2 \in C_0^\infty(\mathbb{R}),\ j = 1, ..., m.$$

It follows

$$A_j \eta = 0, \quad \forall \eta \in C_0^\infty(\mathbb{R}),\ j = 1, ..., m.$$

Using (PE) this implies that $A_j = 0$. □

6. Boundedness of the evolution 1 : abstract setting

The aim of this section is to show that the evolution is bounded outside the complex eigenvalues and the singular points of $\dot{H}$. We assume

(B) $\quad w^{-1} :\ D(h_0) \to D(h_0),\ [-ik, h] \lesssim w^{-1} h_0 w^{-1}$ as quadratic forms on $D(h_0)$.

For $\chi \in C^\infty(\mathbb{R})$ and $\mu > 0$ we put $\chi_\mu(\lambda) = \chi\left(\frac{\lambda}{\mu}\right)$.

Theorem 6.1. *Assume* (A1), (A2), (TE1)-(TE3), (PE), (B)*. Assume furthermore* $\sigma_{pp}^{\mathbb{C}}(\dot{H}) = \emptyset$*. Then*

i) Let $\chi \in C^\infty(\mathbb{R})$ *with* $\chi = 0$ *on* $[-1, 1]$ *and* $\chi = 1$ *outside* $[-2, 2]$*. Then there are* $\mu_0 > 0, C_1 > 0$ *such that for* $\mu \geq \mu_0$*, and* $t \in \mathbb{R}$*:*

$$\|e^{-it\dot{H}} \chi_\mu(\dot{H})u\|_{\dot{\mathcal{E}}} \leq C_1 \|\chi_\mu(\dot{H})u\|_{\dot{\mathcal{E}}},\ u \in \dot{\mathcal{E}}. \tag{6.1}$$

ii) If $\chi \in C_0^\infty(\mathbb{R} \setminus \mathcal{S})$ *then there is* $C_2 > 0$ *such that for all* $u \in \dot{\mathcal{E}}$ *and* $t \in \mathbb{R}$ *we have*

$$\|e^{-it\dot{H}} \chi(\dot{H})u\|_{\dot{\mathcal{E}}} \leq C\|u\|_{\dot{\mathcal{E}}}. \tag{6.2}$$

Remark 6.2. *If* $\sigma_{pp}^{\mathbb{C}}(\dot{H}) \neq \emptyset$*, then the theorem still holds for* $e^{-it\dot{H}}|_{\mathcal{E}_{\mathbb{R}}(\dot{H})}$.

The proof of the Thm. 6.1 is divided into a high (part *i*)) and a low frequency analysis (part *ii*)).

6.1. High frequency analysis.

Lemma 6.3. *Assume* (A1), (A2), (TE1)-(TE3), (PE), (B)*. If* χ *is as in the statement of Thm. 6.1 then for* $\mu > 0$ *sufficiently large we have:*

$$\|(\chi_\mu(\dot{H})u)_0\|_{\mathcal{H}} \lesssim \frac{1}{\mu} \|\chi_\mu(\dot{H})u\|_{\dot{\mathcal{E}}}.$$

Proof. Let $\hat{\chi}$ be as χ with $\hat{\chi}\chi = \chi$. Set $\varphi = \hat{\chi} - 1$ and observe that $\varphi = -1$ on $(-1, 1)$. Let $\tilde{\varphi}$ be some (finite order) almost analytic extension of φ given by ($N \geq 1$):

$$\tilde{\varphi}(x + iy) = \sum_{r=0}^{N} \varphi^{(r)}(x) \frac{(iy)^r}{r!} \tau\left(\frac{y}{\delta\langle x\rangle}\right)$$

with $\tau \in C_0^\infty(\mathbb{R})$, $\tau(s) = 1$ in $|s| \leq 1/2$, $\tau(s) = 0$ in $|s| \geq 1$. Here δ is chosen such that $\dot{R}(z)$ has no poles in $|\mathrm{Im} z| \leq \delta\langle x\rangle$, $x \in \mathrm{supp}\, \varphi$. We compute

$$\begin{aligned}\overline{\partial}\tilde{\varphi}(z) =& \hat{\chi}^{(N+1)}(x) \frac{(iy)^{(N+1)}}{(N+1)!} \tau\left(\frac{y}{\delta\langle x\rangle}\right) + \left(\sum_{r=0}^{N} \varphi^{(r)}(x) \frac{(iy)^r}{r!}\right) \tau'\left(\frac{y}{\delta\langle x\rangle}\right) \left(\frac{i}{\delta\langle x\rangle} + \frac{yx}{\delta\langle x\rangle^2}\right) \\ =:& \tilde{\varphi}_1(x + iy) + \tilde{\varphi}_2(x + iy).\end{aligned}$$

Let $\mu \geq \mu_0 = \max\{(1+\varepsilon)\|k\|_{\mathcal{B}(\mathcal{H})}, \frac{2(1+\varepsilon)}{\delta}\|k\|_{\mathcal{B}(\mathcal{H})}\}$. We then have $\operatorname{supp}\overline{\partial}\tilde{\varphi} \subset K := \{z \in \mathbb{C};\ |\mu z| \geq (1+\varepsilon)\|k\|_{\mathcal{B}(\mathcal{H}))}\} \cap \{z \in \mathbb{C};\ |z| \geq \min\{1, \frac{1}{2}\delta\}\}$. Indeed on $\operatorname{supp}\tilde{\varphi}_1$ we have $|z| \geq 1$ and thus

$$|\mu z| \geq \mu_0 = (1+\varepsilon)\|k\|_{\mathcal{B}(\mathcal{H})}.$$

On $\operatorname{supp}\tilde{\varphi}_2$ we have $|z| \geq \frac{\delta}{2}|z|$ and thus

$$|\mu z| \geq \mu\frac{\delta}{2} \geq (1+\varepsilon)\|k\|_{\mathcal{B}(\mathcal{H})}.$$

Note that

$$1 = -(\chi - 1)(0) = \frac{1}{2\pi i}\int \overline{\partial}\tilde{\varphi}(z)\frac{1}{z}dz \wedge d\overline{z}.$$

We have

$$\begin{aligned}
\hat{\chi}_\mu(\dot{H}) &= \varphi_\mu(\dot{H}) + 1 \\
&= \frac{1}{2\pi i}\int \overline{\partial}\tilde{\varphi}(z)\left(\left(\frac{\dot{H}}{\mu} - z\right)^{-1} + \frac{1}{z}\right)dz \wedge d\overline{z} \\
&= -\frac{1}{2\pi i}\int \overline{\partial}\tilde{\varphi}(z)(\dot{H} - \mu z)^{-1}\frac{\dot{H}}{z}dz \wedge d\overline{z}. \qquad (6.3)
\end{aligned}$$

Let $v^\mu = \chi_\mu(\dot{H})u$. We compute

$$\left((\dot{H} - \mu z)^{-1}\frac{\dot{H}}{z}\begin{pmatrix} v_0^\mu \\ v_1^\mu \end{pmatrix}\right)_0 = \frac{1}{z}p^{-1}(\mu z)(\mu z v_1^\mu + h v_0^\mu).$$

We estimate for $z \in \operatorname{supp}\overline{\partial}\tilde{\varphi}(z)$ using Proposition 2.6:

$$\begin{aligned}
\|p^{-1}(\mu z)z\mu v_1^\mu\|_{\mathcal{H}} &\lesssim \|p^{-1}(\mu z)z\mu(v_1^\mu - k v_0^\mu)\|_{\mathcal{H}} + \|p^{-1}(\mu z)z\mu k v_0^\mu\|_{\mathcal{H}} \\
&\lesssim \frac{1}{|\mathrm{Im} z|\mu}\|(v_1^\mu - k v_0^\mu)\|_{\mathcal{H}} + \frac{1}{\mu|\mathrm{Im} z|}\|v_0^\mu\|_{\mathcal{H}}, \\
\|p^{-1}(\mu z)h v_0^\mu\|_{\mathcal{H}} &\lesssim \|p^{-1}(z)h_0 v_0^\mu\|_{\mathcal{H}} + \|p^{-1}(\mu z)k^2 v_0^\mu\|_{\mathcal{H}} \\
&\lesssim \frac{1}{|\mathrm{Im} z|\mu}\|h_0^{1/2}v_0^\mu\|_{\mathcal{H}} + \frac{1}{\mu^2|\mathrm{Im} z|}\|v_0^\mu\|_{\mathcal{H}} \\
&\lesssim \frac{1}{|\mathrm{Im} z|\mu}\|h_0^{1/2}v_0^\mu\|_{\mathcal{H}} + \frac{1}{\mu^2|\mathrm{Im} z|}\|v_0^\mu\|_{\mathcal{H}}
\end{aligned}$$

Using (6.3) we obtain

$$\|(\chi_\mu(\dot{H}^n)u)_0\|_{\mathcal{H}} \lesssim \frac{1}{\mu}\|\chi_\mu(\dot{H}^n)u\|_{\dot{\mathcal{E}}} + \frac{1}{\mu^2}\|(\chi_\mu(\dot{H}^n)u)_0\|_{\mathcal{H}}.$$

This gives the lemma for μ sufficiently large. □

Corollary 6.4. *Assume* (A1), (A2), (TE1)-(TE3), (PE), (B). *Let χ be as in Thm. 6.1. Then for $\mu > 0$ sufficiently large there exists $\varepsilon > 0$ such that for all $u \in \mathcal{E}$:*

$$\langle \chi_\mu(\dot{H})u, \chi_\mu(\dot{H})u\rangle_0 \geq \varepsilon\|\chi_\mu(\dot{H})u\|^2_{\dot{\mathcal{E}}}.$$

Proof. By Lemma 6.3 we have

$$\begin{aligned}\langle \chi_\mu(\dot H)u, \chi_\mu(\dot H)u\rangle_0 &\geq \|\chi_\mu(\dot H)u\|^2_{\dot{\mathcal{E}}} - 2\|k(\chi_\mu(\dot H)u)_0\|^2_{\mathcal{H}}\\ &\geq (1-\frac{C}{\mu^2})\|(\chi_\mu(\dot H)u)\|^2_{\dot{\mathcal{E}}},\end{aligned}$$

which gives the corollary for μ sufficiently large. □

Corollary 6.5. *Assume* (A1), (A2), (TE1)-(TE3), (PE), (B). *Let χ be as in Thm. 6.1. Then there exists $C_1 > 0$ such that for all $u \in \dot{\mathcal{E}}$, $t \in \mathbb{R}$*

$$\|e^{-it\dot H}\chi_\mu(\dot H)u\|_{\dot{\mathcal{E}}} \leq C_1\|\chi_\mu(\dot H)u\|_{\dot{\mathcal{E}}}.$$

Proof. We use that $\langle e^{-it\dot H}\chi_\mu(\dot H)u, e^{-it\dot H}\chi_\mu(\dot H)u\rangle_0$ is conserved. By Corollary 6.4 we have

$$\begin{aligned}\|e^{-itH}\chi_\mu(\dot H)u\|^2_{\dot{\mathcal{E}}} &\lesssim \langle e^{-it\dot H}\chi_\mu(\dot H)u, e^{-it\dot H}\chi_\mu(\dot H)u\rangle_0\\ &= \langle \chi_\mu(\dot H)u, \chi_\mu(\dot H)u\rangle_0 \lesssim \|\chi_\mu(\dot H)u\|^2_{\dot{\mathcal{E}}},\end{aligned}$$

which finishes the proof. □

6.2. **Low frequency analysis.** Part *ii*) of Thm. 6.1 follows from the following

Lemma 6.6. *Assume* (A1), (A2), (TE1)-(TE3), (PE), (B). *Let $\chi \in C_0^\infty(\mathbb{R}\setminus\mathcal{S})$. Then there exists $C > 0$ such that:*

$$\|e^{-it\dot H}\chi(\dot H)u\|_{\dot{\mathcal{E}}} \leq C\|u\|_{\dot{\mathcal{E}}},\ u \in \dot{\mathcal{E}},\ t \in \mathbb{R}. \tag{6.4}$$

Proof. Let $u \in \dot{\mathcal{E}}$. Let

$$\psi(t) := (\psi_0(t), \psi_1(t)) := e^{-itk}\begin{pmatrix} 1 & 0\\ -k & 1\end{pmatrix} e^{-it\dot H}\chi(\dot H)u.$$

Note that

$$\|e^{-it\dot H}\chi(\dot H)u\|^2_{\dot{\mathcal{E}}} = \|\psi_1\|^2 + (h(t)\psi_0(t)|\psi_0(t)) =: \|\psi(t)\|^2_{\mathcal{E}(t)}$$

with $h(t) = e^{-itk}(h+k^2)e^{itk}$ and that $\psi(t)$ solves the wave equation

$$(\partial_t^2 + h(t))\psi_0(t) = 0, \quad \psi_1(t) = -i\partial_t\psi_0(t).$$

Thus

$$\frac{d}{dt}\|e^{-it\dot H}\chi(\dot H)u\|^2_{\dot{\mathcal{E}}} = (h'(t)\psi_0(t)|\psi_0(t)),$$

where

$$h'(t) = e^{-itk}[-ik, h]e^{ikt} \lesssim w^{-1}e^{-ikt}h_0e^{ikt}w^{-1},$$

using (B). Therefore :

$$\frac{d}{dt}\|e^{-it\dot H}\chi(\dot H)u\|^2_{\dot{\mathcal{E}}} \lesssim \|w^{-1}\psi(t)\|^2_{\mathcal{E}(t)} \lesssim \|w^{-1}e^{-it\dot H}\chi(\dot H)u\|^2_{\dot{\mathcal{E}}}, \tag{6.5}$$

where we have used that w^{-1} commutes with $e^{-itk}\begin{pmatrix} 1 & 0\\ k & 1\end{pmatrix}$ (see (TE1)). Integrating (6.5) we obtain :

$$\|e^{-it\dot H}\chi(\dot H)u\|^2_{\dot{\mathcal{E}}} \lesssim \|u\|^2_{\dot{\mathcal{E}}} + \int_0^t \|w^{-1}e^{-it\dot H}\chi(\dot H)u\|^2_{\dot{\mathcal{E}}}dt \lesssim \|u\|^2_{\dot{\mathcal{E}}},$$

by Prop.5.8. □

We end this section by proving a weak convergence result, which will be important in Sect. 7.

Lemma 6.7. *Assume* (A1)-(A2), (TE1)-(TE3), (PE)-(B). *Then*

$$e^{-it\dot{H}}\chi(\dot{H}) \rightharpoonup 0, \ \forall\ \chi \in C_0^\infty(\mathbb{R}\backslash\mathcal{S}).$$

Proof. Since $\mathrm{e}^{-it\dot{H}}\chi(\dot{H})$ is uniformly bounded in t by Thm. 6.3, it suffices to prove that $\langle v|e^{-it\dot{H}}\chi(\dot{H})u\rangle_{\dot{\mathcal{E}}} \to 0$ for u, v in a dense subspace of $\dot{\mathcal{E}}$, where $\langle\cdot|\cdot\rangle_{\dot{\mathcal{E}}}$ is the scalar product associated to the norm of $\dot{\mathcal{E}}$. By (PE) the space $\{u \in \dot{\mathcal{E}}\ :\ u = \chi(x)u,\ \chi \in C_0^\infty(\mathbb{R})\}$ is dense in $\dot{\mathcal{E}}$. For such u, v the convergence to 0 follows from Prop. 5.7. □

7. Asymptotic completeness 1: abstract setting

In this section we prove existence and completeness of wave operators, comparing the full dynamics $\mathrm{e}^{-it\dot{H}}$ with the two asymptotic dynamics $\mathrm{e}^{-it\dot{H}_\pm}$, for energies away from the set $\mathcal{S}$ of singular points. We first define the spaces of *scattering states*.

Definition 7.1. *We call* $\chi \in C^\infty(\mathbb{R})$ *an* admissible *cut-off function for* $\dot{H}$ *if*

- $\chi = 0$ *in a neighborhood of* $\mathcal{S}$ *and*
- $\chi = 0$ *or* $\chi = 1$ *on* $\mathbb{R}\backslash]-R, R[$ *for some* $R > 0$.

We note $\mathcal{C}^H$ *the set of all admissible cut-offs for* $\dot{H}$.

Definition 7.2. *The spaces of* scattering states *are defined as*

$$\begin{array}{rcl} \dot{\mathcal{E}}_{scatt} & := & \{\chi(\dot{H})u;\ u \in \dot{\mathcal{E}},\ \chi \in \mathcal{C}^H\}, \\ \dot{\mathcal{E}}_{scatt\pm} & := & \{\chi(\dot{H}_\pm)u;\ u \in \dot{\mathcal{E}}_\pm,\ \chi \in \mathcal{C}^H\}. \end{array}$$

We will need the following three lemmas :

Lemma 7.3. *Assume* (A1)-(A2) *and* (TE1)-(TE3). *Then* $w[\dot{H}, i_\pm]w \in \mathcal{B}(\dot{\mathcal{E}}; \dot{\mathcal{E}}_\pm)$.

Proof. We have

$$w[\dot{H}, i_\pm]w = \begin{pmatrix} 0 & 0 \\ w[h, i_\pm]w & 0 \end{pmatrix} \in B(\dot{\mathcal{E}}, \dot{\mathcal{E}}_\pm),$$

by hypothesis (TE3)*e)*. □

Lemma 7.4. *Assume* (A1)-(A2) *and* (TE1)-(TE3).

i) *Let* $\chi \in C_0^\infty(\mathbb{R})$. *Then*

$$i_\pm\chi(\dot{H}_\pm) - \chi(\dot{H})i_\pm \in \mathcal{B}_\infty(\dot{\mathcal{E}}_\pm; \dot{\mathcal{E}}).$$

ii) *Let* $\chi \in C^\infty(\mathbb{R})$ *such that* $\chi = 1$ *outside* $]-R, R[$ *for some* $R > 0$. *Then*

$$i_\pm\chi(\dot{H}_\pm) - \chi(\dot{H})i_\pm \in \mathcal{B}_\infty(\dot{\mathcal{E}}_\pm; \dot{\mathcal{E}}).$$

Proof. Note first that *ii*) follows from *i*), replacing χ by $1-\chi$. We therefore only have to prove *i*). Let $\tilde{\chi}$ be an almost analytic extension of χ such that $\operatorname{supp}\tilde{\chi}$ doesn't intersect the complex poles of $\dot{R}(z)$. We have

$$i_{\pm}\chi(\dot{H}_{\pm})-\chi(\dot{H})i_{\pm}=\frac{1}{2\pi i}\int\overline{\partial}\tilde{\chi}(z)\dot{R}(z)[\dot{H},i_{\pm}]\dot{R}_{\pm}(z)dz\wedge d\overline{z}.$$

By hypotheses (TE3)*b)* and (TE3)*e)* we have

$$[\dot{H},i_{\pm}]\dot{R}_{\pm}(z)\in\mathcal{B}_{\infty}(\dot{\mathcal{E}}_{\pm};\dot{\mathcal{E}}).$$

Then we apply the estimates in Lemma 4.4. □

Theorem 7.5. *Assume* (A1), (A2), (TE1)-(TE3), (PE)-(B).

(i) For all $\varphi^{\pm}\in\dot{\mathcal{E}}_{scatt\pm}$ *there exist* $\psi^{\pm}\in\dot{\mathcal{E}}_{scatt}$ *such that*

$$e^{-it\dot{H}}\psi^{\pm}-i_{\pm}e^{-it\dot{H}_{\pm}}\varphi^{\pm}\to 0,\ t\to\infty\quad in\quad \dot{\mathcal{E}}.$$

(ii) For all $\psi^{\pm}\in\dot{\mathcal{E}}_{scatt}$ *there exist* $\varphi^{\pm}\in\dot{\mathcal{E}}_{scatt\pm}$ *such that*

$$e^{-it\dot{H}_{\pm}}\varphi^{\pm}-i_{\pm}e^{-it\dot{H}}\psi^{\pm}\to 0,\ t\to\infty\quad in\quad \dot{\mathcal{E}}_{\pm}.$$

Proof. Let $\chi\in\mathcal{C}^{H}$. We only prove (i), the proof of (ii) being analogous.
We first show that the limit

$$W^{\pm}\varphi:=\lim_{t\to\infty}e^{it\dot{H}}\chi(\dot{H})i_{\pm}e^{-it\dot{H}_{\pm}}\chi(\dot{H}_{\pm})\varphi \tag{7.1}$$

exists for all $\varphi\in\dot{\mathcal{E}}_{\pm}$. We first treat the case $\chi=0$ on $\mathbb{R}\backslash]-R,R[$. Using Thm. 6.1 *i)*, Lemma 4.2 and the fact that $\dot{H}_{\pm}$ are selfadjoint, we obtain:

$$\|e^{it\dot{H}}\chi(\dot{H})i_{\pm}e^{-it\dot{H}_{\pm}}\chi(\dot{H}_{\pm})\varphi\|_{\dot{\mathcal{E}}}\lesssim\|\varphi\|_{\dot{\mathcal{E}}_{\pm}}. \tag{7.2}$$

By (7.2) and assumption (PE) we may assume that $\varphi\in D(w)$. We compute

$$\frac{d}{dt}e^{it\dot{H}}\chi(\dot{H})i_{\pm}e^{-it\dot{H}_{\pm}}\chi(\dot{H}_{\pm})=e^{it\dot{H}}\chi(\dot{H})[\dot{H},i_{\pm}]e^{-it\dot{H}_{\pm}}\chi(\dot{H}_{\pm}). \tag{7.3}$$

Integrating (7.3) and using Lemma 7.3 and Proposition 5.7 we obtain:

$$\begin{aligned}&\|e^{it\dot{H}}\chi(\dot{H})i_{\pm}e^{-it\dot{H}_{\pm}}\chi(\dot{H}_{\pm})-e^{is\dot{H}}\chi(\dot{H})i_{\pm}e^{-is\dot{H}_{\pm}}\chi(\dot{H}_{\pm})\varphi\|\\ \lesssim&\int_{s}^{t}\|w^{-1}e^{-it'\dot{H}_{\pm}}\chi(\dot{H}_{\pm})\varphi\|_{\dot{\mathcal{E}}_{\pm}}\lesssim\int_{s}^{t}\langle t'\rangle^{-2}dt'\|w\varphi\|_{\dot{\mathcal{E}}_{\pm}}\to 0,\ s,t\to\infty.\end{aligned}$$

This gives the existence of the limit (7.1). Let now $\chi=1$ on $\mathbb{R}\backslash]-R,R[$. Let $\hat{\chi}\in C_0^{\infty}(\mathbb{R})$, $\hat{\chi}=1$ in a neighborhood of 0. Using that $e^{it\dot{H}}\chi(\dot{H})$ is uniformly bounded by Thm. 6.1 *ii)*, Lemma 7.4 and Lemma 5.11 we see that

$$\begin{aligned}&\text{s-}\lim_{t\to\infty}e^{it\dot{H}}\chi(\dot{H})i_{\pm}e^{-it\dot{H}_{\pm}}\hat{\chi}^{2}\left(L^{-1}\dot{H}_{\pm}\right)\chi(\dot{H}_{\pm})\\ =&\text{s-}\lim_{t\to\infty}e^{it\dot{H}}\chi(\dot{H})\hat{\chi}\left(L^{-1}\dot{H}\right)i_{\pm}e^{-it\dot{H}_{\pm}}\hat{\chi}\left(L^{-1}\dot{H}_{\pm}\right)\chi(\dot{H}_{\pm})\end{aligned}$$

exists, since $\hat{\chi}(L^{-1}\cdot)$ is compactly supported. Let $\epsilon > 0$. We estimate

$$\begin{aligned}
&\|e^{it\dot{H}}\chi(\dot{H})i_\pm e^{-it\dot{H}_\pm}\chi(\dot{H}_\pm)\varphi^\pm - e^{is\dot{H}}\chi(\dot{H})i_\pm e^{-is\dot{H}_\pm}\chi(\dot{H}_\pm)\varphi^\pm\|_{\dot{\mathcal{E}}} \\
&\le \left\|e^{it\dot{H}}\chi(\dot{H})i_\pm e^{-it\dot{H}_\pm}\hat{\chi}^2(L^{-1}\dot{H}_\pm)\chi(\dot{H}_\pm)\varphi^\pm - e^{is\dot{H}}\chi(\dot{H})i_\pm e^{-is\dot{H}_\pm}\hat{\chi}^2(L^{-1}\dot{H}_\pm)\chi(\dot{H}_\pm)\varphi^\pm\right\|_{\dot{\mathcal{E}}} \\
&\quad + 2\left\|(1-\hat{\chi}^2(L^{-1}\dot{H}_\pm))\varphi^\pm\right\|_{\dot{\mathcal{E}}} < \epsilon,
\end{aligned}$$

if we choose first L and then t, s large enough. This shows the existence of the limit (7.1) if $\chi = 1$ on $\mathbb{R}\backslash] - R, R[$.

Let for $\phi^\pm \in \dot{\mathcal{E}}_\pm$

$$\psi_t^\pm = e^{it\dot{H}}\chi(\dot{H})i_\pm e^{-it\dot{H}_\pm}\chi(\dot{H}_\pm)\phi^\pm,\ \psi^\pm = \lim_{t\to\infty}\psi_t^\pm.$$

Let us write

$$\psi_t^\pm = \psi^\pm + r(t),\ r(t) \to 0,\ t \to \infty.$$

Let $\tilde{\chi} \in \mathcal{C}^H$ with $\tilde{\chi}\chi = \chi$. We clearly have $\tilde{\chi}(\dot{H})\psi_t^\pm = \psi_t^\pm$ and thus

$$\tilde{\chi}(\dot{H})\psi^\pm + \tilde{\chi}(\dot{H})r(t) = \psi^\pm + r(t).$$

Taking the limit $t \to \infty$ we find :

$$\tilde{\chi}(\dot{H})\psi^\pm = \psi^\pm \quad \text{in particular} \quad \psi^\pm \in \dot{\mathcal{E}}_{scatt},$$

hence $\mathrm{e}^{-it\dot{H}}\psi^\pm$ is uniformly bounded by Thm. 6.1. It follows

$$e^{-it\dot{H}}\psi^\pm - \chi(\dot{H})i_\pm e^{-it\dot{H}_\pm}\chi(\dot{H}_\pm)\phi^\pm \to 0.$$

Applying Lemma 7.4 we find

$$e^{-it\dot{H}}\psi^\pm - i_\pm e^{-it\dot{H}_\pm}\chi^2(\dot{H}_\pm)\phi^\pm \to 0,\ t \to \infty. \tag{7.4}$$

Applying once more a density argument we obtain *i)*. □

8. Geometric setting

We describe in this section an intermediate geometric framework corresponding to our abstract framework. The main example will of course be the Klein-Gordon equation on the Kerr-De Sitter spacetime. The main part of the section will consist in checking that the geometric hypotheses introduced below imply the abstract hypotheses in Sects. 5, 7.

We consider a d dimensional manifold of the form $\mathcal{M} =]r_-, r_+[_r \times S_\omega^{d-1}$. Let

$$P = \sum_{ij=1}^{d-1} D_i^*\alpha_{ij}(\omega)D_j \ge 0$$

be a symmetric elliptic operator on $L^2(S_\omega^{d-1}; d\omega)$. Then $(P, H^2(S_\omega^{d-1}; d\omega))$ is selfadjoint. We assume that for a suitable choice of θ_1, $L^2(S_\omega^{d-1}; d\omega)$ possesses a basis of eigenfunctions of D_{θ_1}. Let Y^n be the eigenspace corresponding to the eigenvalue n. Then we have

$$L^2(S_\omega^{d-1}; d\omega) = \oplus_{n\in\mathbb{Z}}Y^n.$$

Our first assumption is

$$[P, D_{\theta_1}] = 0,\ \text{ie } \alpha_{ij} \text{ are independent on } \theta_1. \tag{G1}$$

Let $q(r) := \sqrt{(r_+ - r)(r - r_-)}$. We will need the following function spaces

$$T^\sigma = \{f \in C^\infty(\mathcal{M}) : \quad \partial_r^\alpha \partial_\omega^\beta f \in \mathcal{O}(q(r)^{\sigma-2\alpha})\}.$$

Let $i_\pm \in C^\infty([r_-, r_+])$, $i_- = 0$ in a neighborhood of r_+, $i_+ = 0$ in a neighborhood of r_- and $i_-^2 + i_+^2 = 1$.

8.1. **Separable Hamiltonians.** We will consider a Hamiltonian $h_{0,s}$ of the following form:

$$h_{0,s} = \alpha_1 D_r \alpha_2^2 D_r \alpha_1 + \alpha_3^2 P + \alpha_4^2. \tag{8.1}$$

Here α_i, $1 \le i \le 4$ are smooth functions depending only on r. We suppose that there exist $\alpha_j^\pm \in \mathbb{R}$, $1 \le j \le 4$ such that for some $\delta > 0$

$$\alpha_j - q(r)(i_- \alpha_j^- + i_+ \alpha_j^+) \in T^{1+\delta}, \ \alpha_j \gtrsim q(r). \tag{G2}$$

Note that (G2) implies

$$\alpha_j \in T^1, \ \alpha_j \lesssim q(r). \tag{8.2}$$

We will also need an operator k_s of the form

$$k_s = k_{s,r} D_{\theta_1} + k_{s,v}. \tag{8.3}$$

Here $k_{s,r}$ and $k_{s,v}$ are smooth functions depending only on r. We suppose that there exist $k_{s,v}^-$, $k_{s,r}^- \in \mathbb{R}$ such that for some $\delta > 0$

$$\left\{ \begin{array}{rcl} i_+ k_{s,r},\, i_+ k_{s,v} & \in & T^2, \\ i_-(k_{s,r} - k_{s,r}^-) & \in & T^2, \\ i_-(k_{s,v} - k_{s,v}^-) & \in & T^2. \end{array} \right. \tag{G3}$$

We put:

$$h_s = h_{0,s} - k_s^2.$$

The associated separable Klein-Gordon equation is:

$$(\partial_t^2 - 2ik_s \partial_t + h_s)u = 0. \tag{8.4}$$

We introduce the Hilbert spaces $\mathcal{H} = L^2(]r_-, r_+[_r \times S_\omega^{d-1}; dr d\omega)$ and $\mathcal{H}^n = \mathcal{H} \cap Y^n$.

8.2. **Perturbed Hamiltonian.** We consider a perturbation of (8.4) of the form

$$(\partial_t^2 - 2ik\partial_t + h)u = 0. \tag{8.5}$$

We first fix an operator h_0 of the form :

$$\begin{array}{rl} h_0|_{C_0^\infty(\mathcal{M})} = & h_{0,s} + \sum_{i,j\in\{1,\dots,d-1\}} D_i^* g^{ij} D_j + \sum_{i\in\{1,\dots,d-1\}} (g^i D_i + D_i^* \overline{g} i) \\ & + D_r g^{rr} D_r + g^r D_r + D_r \overline{g}^r + f \\ =: & h_{0,s} + h_p. \end{array} \tag{8.6}$$

We assume

$$\text{The functions } g^{ij},\ g^i,\ g^{rr},\ g^r,\ f \text{ are independent of } \theta_1, \tag{G4}$$

and

$$\left\{ \begin{array}{rcl} h_0 & \gtrsim & \alpha_1(r)(D_r q^2(r) D_r + P + 1)\alpha_1(r), \\ h_{0,s} & \gtrsim & \alpha_1(r)(D_r q^2(r) D_r + P + 1)\alpha_1(r). \end{array} \right. \tag{G5}$$

Note that (G5) implies (A1). $(h_0, C_0^\infty(\mathcal{M}))$ is symmetric, we denote by $(h_0, D(h_0))$ its Friedrichs extension. The asymptotic behavior of the various coefficients is assumed to be as follows: we require that there exists $\delta > 0$ such that

$$\text{(G6)} \qquad \begin{aligned} &g^{ij} \in T^{2+\delta},\ g^{rr} \in T^{4+\delta}, \\ &g^{r} \in T^{2+\delta},\ g^{i} \in T^{2},\ f \in T^{2}. \end{aligned}$$

We also set

$$k := k_r D_{\theta_1} + k_v,\ k_r = k_{s,r} + k_{p,r},\ k_v = k_{s,v} + k_{p,v}$$

and suppose

$$\text{(G7)} \qquad k_{p,r},\ k_{p,v} \in T^2.$$

Finally we set

$$h := h_0 - k^2.$$

All operators have natural restrictions to the space $\mathcal{H}^n$, which we denote by a subscript n, for example h_0^n is the restriction of h_0 to $\mathcal{H}^n$. The operator k^n is bounded and $(h^n, D(h_0^n))$ is selfadjoint. We will check the hypotheses for the restrictions of the operators to $\mathcal{H}^n$ and the corresponding energy spaces. We will drop the index n in the following.

8.3. **Asymptotic Hamiltonians.** Let us first introduce the change of variables given by

$$\frac{dx}{dr} = \alpha_1^{-2}(r).$$

Note that there is a freedom in the choice of the integration constant. The choice of this constant is however not important for what follows. For $r \to r_-$ we find

$$x(r) - x(r_-) = \int_{r_-}^{r} \frac{1}{\alpha_-^2 q^2} dr + \int_{r_-}^{r} h(r) dr$$

with $h(r) \in \mathcal{O}((r-r_-)^{-1+\delta})$. Here we have used (G2). Recalling that $q(r) = \sqrt{(r_+ - r)(r - r_-)}$ we find for r close to r_-

$$x(r) - x(r_-) \geq \frac{1}{\kappa_-} \ln(r - r_-) - C$$

with $\kappa_- = (\alpha_-)^2 (r_+ - r_-)$. It follows

$$r - r_- \lesssim e^{\kappa_- x}, \quad r \to r_-.$$

In a similar way we obtain

$$r_+ - r \lesssim e^{-\kappa_+ x}, \quad r \to r_+,$$

where $\kappa_+ = (\alpha_+)^2 (r_+ - r_-)$. Using that $\partial_x = \alpha_1^{-2}(r) \partial_r$ we find :

$$\begin{aligned} f(r) \quad &\in \quad T^\sigma \quad \text{iff} \quad f(r(x)) \in T_x^\sigma \quad \text{for} \\ T_x^\sigma \quad &= \quad \left\{ f \in C^\infty(\mathbb{R} \times S_\omega^{d-1});\ \partial_x^\alpha \partial_\omega^\beta f \in \left\{ \begin{array}{l} \mathcal{O}(e^{\frac{\sigma\kappa_-}{2} x}),\ x \to -\infty, \\ \mathcal{O}(e^{-\frac{\sigma\kappa_+}{2} x}), \quad x \to \infty. \end{array} \right\} \right\}. \end{aligned}$$

This change of variables gives rise to the unitary transformation

$$\mathcal{U}_1 : \begin{array}{rcl} L^2(]r_-, r_+[_r \times S_\omega^{d-1}; dr d\omega) & \to & L^2(\mathbb{R} \times S_\omega^{d-1}; dx d\omega) =: \mathcal{H}_1 \\ v(r, \omega) & \mapsto & \alpha_1(r(x)) v(x, \omega). \end{array}$$

We put

$$\mathcal{E}_{\pm,1} = (\mathcal{U}_1 \oplus \mathcal{U}_1) \mathcal{E}_\pm,\ h_\pm^1 := \mathcal{U}_1 h_\pm \mathcal{U}_1^{-1},\ k_\pm^1 = \mathcal{U}_1 k_\pm \mathcal{U}_1^{-1}.$$

We compute

$$\begin{aligned} h_0^1 = \mathcal{U}_1 h_0 \mathcal{U}_1^{-1} = \mathcal{U}_1 h_{0,s}\mathcal{U}_1^{-1} &+ \sum_{i,j\in\{1,\dots,d-1\}} D_i^* g^{ij} D_j + \sum_{i\in\{1,\dots,d-1\}} (g^i D_i + D_i^*\overline{g}i) \\ &+ \alpha_1^{-1}(r) D_x g^{rr}\alpha_1^{-2}(r) D_x \alpha_1^{-1}(r) + \alpha_1^{-1}(r) g^r D_x \alpha_1^{-1}(r) + \alpha_1^{-1}(r) D_x \overline{g}^r \alpha_1^{-1}(r) + f, \\ \mathcal{U}_1 h_{0,s}\mathcal{U}_1 =& D_x \alpha_2^2 \alpha_1^{-2} D_x + \alpha_3^2 P + \alpha_4^2. \end{aligned}$$

We will often drop the exponent 1 when it is clear which coordinate system is used.

8.3.1. *Asymptotic Hamiltonians.* The asymptotic Hamiltonians are now constructed as in Subsect. 4.2: we set

$$\ell := nk_{s,r}^- + k_{s,v}^-,$$

and define $k_\pm$, h_+, $\tilde{h}_-$ as in (4.1), (4.2). Recall that since l is fixed, these operators depend only on a cutoff scale R. The following hypothesis is the analog of (TE2):

(G8) $\quad (h_+, k_+),\ (\tilde{h}_-, k_- - l)$ satisfy (G5) with h_0 replaced by $h_+,\ \tilde{h}_-$.

8.4. **Meromorphic extensions.** In this subsection we will check that $h_+, \tilde{h}_-$ satisfy (ME2). To do so we use a result of Mazzeo and Melrose about the meromorphic extension of the truncated resolvent for the Laplace operator on asymptotically hyperbolic manifolds, see [24]. We start by briefly recalling this result.

8.4.1. *A result of Mazzeo-Melrose.* Let Y be a compact $n-$ dimensional manifold with boundary given by the defining function y:

$$\partial Y = \{y = 0\},\ dy|_{\partial Y} \neq 0,\ y|_{Y^0} > 0.$$

Let g be a complete metric on Y of the form

$$g = \frac{h}{y^2}, \tag{8.7}$$

where h is a C^∞ metric on Y. One is usually interested in the study of the Laplace-Beltrami operator Δ_g. We have to consider slightly more general operators. Let

$$\mathcal{V}_0(Y) = \{V \in C^\infty(Y; TY);\ V|_{\partial Y} = 0\}$$

the space of vector fields vanishing on the boundary. In local coordinates (y, x) near ∂Y the vector fields $y\partial_y$, $y\partial_{x_j}$ span $\mathcal{V}_0(Y)$.

We now need the definition of the *normal operator*. For $p \in \partial Y$ the tangent space, T_pY, is divided into two half-spaces by the hypersurface $T_p\partial Y$. We will denote by Y_p the half space on the "y" side (that is spanned by $T_p\partial Y$ and the inward normal vector at p). Then any smooth coefficient polynomial in $\mathcal{V}_0(Y)$ Q defines a natural constant coefficient operator on Y_p:

$$N_p(Q)u = \lim_{r\to 0} R_r^* f^* Q (f^{-1})^* R_{\frac{1}{r}}^* u, \tag{8.8}$$

where $u \in C^\infty(Y_p)$, R_r is the natural $\mathbb{R}_+$ action on $Y_p \simeq N_+T_p\partial Y$ given by the multiplication by r on the fibers and f is a local diffeomorphism from $\Omega \subset Y$, $p \in \Omega$:

$$f : \Omega \to \Omega',\ \Omega' \subset T_pY,\ f(p) = 0,\ df_p = I,\ f(\partial Y) \subset T_p\partial Y.$$

The definition is independent of f. The normal operator freezes the coefficients at a point p, one obtains a polynomial in the elements of $\mathcal{V}_0(Y_p)$. The following result is implicit in [24]. We use here the formulation in [26].

Proposition 8.1 (Mazzeo-Melrose 87)**.** *Let Q be a second order differential operator on Y which is a polynomial in $\mathcal{V}_0(Y)$ with coefficients in $C^\infty(Y)$. Assume that*

i) *the principal part of Q is an elliptic polynomial in the elements of $\mathcal{V}_0(Y)$ uniformly on Y,*
ii) *for every $p \in \partial Y$ the normal operator of Q defined by* (8.8) *is given by*

$$\begin{aligned} N_p(Q) &= -K\left[z_1^2 D_{z_1}^2 + i(n-2)z_1 D_{z_1} + z_1^2 \sum_{i,j=2}^n h_{ij}(p) D_{z_i} D_{z_j} - \left(\frac{n-1}{2}\right)^2\right], \\ Y_p &= \{z \in \mathbb{R}^n;\ z_1 \geq 0\}, \quad [h_{ij}] \geq C\mathbb{1}, C > 0, \end{aligned}$$

where $K < 0$ is constant on the components of ∂Y.

Then for any metric g of the form (8.7)

$$R_Q(z) = (Q - z^2)^{-1} : L^2(Y, dvol_g) \to L^2(Y, dvol_g)$$

is holomorphic in $\{\mathrm{Im} z \gg 1\}$*. For $N > 0$ the operator $y^N R_Q(z) y^N$ extends to a meromorphic operator in* $\{\mathrm{Im} z > -\delta\}$ *for some $\delta > 0$.*

Remark 8.2. *The width of the strip in which one can extend the truncated resolvent is of order N if one removes some special points along the imaginary axis. At these points which are given by $(-K)^{1/2}(-i)\left(\frac{2k-1}{2}\right)^{1/2}$, $k \in \mathbb{N}$ essential singularities might occur. If the operator is the Laplacian associated to an asymptotic hyperbolic metric and the metric is even, then no essential singularities appear, see* [18] *for a detailed discussion of these questions. In our case it is sufficient to know that there exists a meromorphic extension in some strip and we won't study the type of singularities.*

8.4.2. *Meromorphic extensions of the resolvents of $h_+, \tilde{h}_-$.*

Lemma 8.3. *Assume hypotheses* (G1)—(G8)*. Then (h_+, k_+, w) and $(\tilde{h}_-, k_- - \ell, w)$ satisfy* (ME1)*,* (ME2) *for $w = q(r)^{-1}$.*

Proof. We will show that $w^{-\epsilon}(h_\pm^n - z^2)^{-1} w^{-\epsilon}$ has a meromorphic extension to a strip $\{\mathrm{Im} z > -\delta_\epsilon\}$, $\delta_\epsilon > 0$. Let us start with h_+.

We want to apply Prop. 8.1, for $Y = \overline{\mathcal{M}} = [r_-, r_+] \times S^{d-1}$. The principal part of h_+^n is an elliptic polynomial in the elements of $\mathcal{V}_0(Y)$ and hypothesis $i)$ of Prop. 8.1 is fulfilled. Near each boundary component we put $z = r - r_-$ and $z = r_+ - r$ resp. We change the C^∞ structure on Y (as a manifold with boundary) and allow a new smooth coordinate $y = \sqrt{z}$. We will denote the new manifold by $Y_{1/2}$ and think of $Y_{1/2}$ as a conformally compact manifold in the sense of having a metric of the form (8.7).

The operator h_+^n becomes near $r = r_-$

$$\begin{aligned} h_+ &= \frac{1}{4}(\alpha_1^+)^2(\alpha_2^+)^2(r_+ - r_-)^4 D_y y D_y y + (\alpha_3^+)^2(r_+ - r_-)^2 y^2 \sum_{i,j=1}^{d-1} D_i^* \alpha_{ij} D_j \\ &- (k_{s,r}^+ n + k_{s,v}^+)^2 + \mathcal{O}(y^\delta)(D_y y)^2 + \mathcal{O}(y^\delta) y^2 \sum_{i,j=1}^{d-1} D_i^* \alpha_{ij} D_j + \mathcal{O}(y^\delta). \end{aligned}$$

We now conjugate h_+ by a weight function (see [26]) and set:

$$Q = ((1-\chi) + \chi y^2) h_+ ((1-\chi) + \chi y^2)^{-1}, \tag{8.9}$$

where $\chi \in C^\infty(Y)$, $\chi = 1$ for $y < \epsilon < 1/2$ and $\chi = 0$ for $y > 2\epsilon$. It follows that the normal operator becomes

$$\begin{aligned} N_p(Q) &= \frac{1}{4}(\alpha_1^+)^2(\alpha_2^+)^2(r_+ - r_-)^4 \left(y^2 \left(D_y^2 + \frac{4(\alpha_3^+)^2}{(\alpha_1^+)^2(\alpha_2^+)^2(r_+ - r_-)^2} \sum_{i,j=1}^{d-1} D_i^* \alpha_{ij}(p) D_j \right) \right. \\ &+ \left. i y D_y - 1 - \frac{4(k_{s,r}^+ n + k_{s,v}^+)^2}{(\alpha_1^+)^2(\alpha_2^+)^2(r_+ - r_-)^4} \right). \end{aligned}$$

This operator is shifted with respect to the model operator in Proposition 8.1 and the points where essential singularities may occur are now given by $z^2 = (-K)\left(\beta - \left(\frac{1-2k}{2}\right)^2\right)$, $k \in \mathbb{N}$, where $-K = \frac{1}{4}(\alpha_1^+)^2(\alpha_2^+)^2(r_+ - r_-)^4$, $\quad \beta = -\frac{4(k_{s,r}^- n + k_{s,v})^2}{(\alpha_1^+)^2(\alpha^+)^2(r_+ - r_-)^4}$. As β is negative all these points have strictly negative imaginary part. Hence we obtain a meromorphic continuation of $w^{-\epsilon}(Q - z^2)^{-1} w^{-\epsilon}$, where Q is given by (8.9). Since for $\mathrm{Im} z \gg 0$ we have

$$(h_+ - z^2)^{-1} = ((1-\chi) + \chi y^2)^{-1}(Q - z^2)^{-1}((1-\chi) + \chi y),$$

we obtain a meromorphic continuation of $w^{-\epsilon}(h_+ - z^2)^{-1} w^{-\epsilon}$. The proofs near $r = r_+$ and for $\tilde{h}_-$ are similar. □

8.5. **Verification of the abstract hypotheses.**

Proposition 8.4. *Assume hypotheses* (G1)—(G8). *Then conditions* (A1)-(A2), (TE1)-(TE3), (PE)-(B) *are satisfied*

The rest of the subsection is devoted to the proof of Prop. 8.4. We start by some preparations.

8.5.1. *Some useful facts.* By (G5) we have the following estimates

$$\|q(r) D_r \alpha_1(r) u\| \lesssim \|h_0^{1/2} u\|, \tag{8.10}$$

$$\|q(r) D_j u\| \lesssim \|h_0^{1/2} u\|,\ j = 1, ..., d-1, \tag{8.11}$$

$$\|q(r) u\| \lesssim \|h_0^{1/2} u\|. \tag{8.12}$$

The estimates (8.10)-(8.12) also hold with h_0 replaced by h_+ or $\tilde{h}_-$. We will also need the following Hardy type estimate.

Lemma 8.5. *We have*

$$
\begin{aligned}
&i)\ \|\langle x(r)\rangle^{-1}u\|_{\mathcal{H}} \lesssim \|h_0^{1/2}u\|_{\mathcal{H}},\\
&ii)\ \|fu\|_{\mathcal{H}} \lesssim \|h_0^{1/2}u\|_{\mathcal{H}},\ f\in T^{\delta},\ \delta>0.
\end{aligned}
$$

Proof. Since $\langle x(r)\rangle \sim |\ln(r-r_\pm)|$ when $r\to r_\pm$, *ii)* follows from *i)*. We recall a version of Hardy's inequality:

$$
\int_0^\infty |v(x)|^2x^{-2}dt \le 4\int_0^\infty |v'(x)|^2dx,\ v\in C_0^\infty(\mathbb{R}\backslash\{0\}). \tag{8.13}
$$

Let $\chi_1\in C_0^\infty(\mathbb{R})$, $\chi_1(0)=1$ and $\chi_2\in C^\infty(\mathbb{R})$ with $\chi_1+\chi_2=1$. We have

$$
\|\langle x\rangle^{-1}\chi_1u\|^2_{\mathcal{H}_1} \lesssim \|\chi_1u\|^2_{\mathcal{H}_1} \lesssim ((-\partial_x^2+\alpha_1^2)u|u)_{\mathcal{H}_1}
$$

because $\alpha_1^2 \gtrsim \chi_1^2$. Now applying (8.13) to χ_2u gives :

$$
\begin{aligned}
\|\langle x\rangle^{-1}\chi_2u\|^2_{\mathcal{H}_1} &\lesssim \int_{\mathbb{R}\times S^2}|\partial_x(\chi_2u)|^2dxd\omega \lesssim \int_{\mathbb{R}\times S^2}(\chi_2')^2|u|^2dxd\omega + \int_{\mathbb{R}\times S^2}\chi_2^2|u'|^2dxd\omega\\
&\lesssim ((-\partial_x^2+\alpha_1^2)u|u)_{\mathcal{H}_1}
\end{aligned}
$$

because $(\chi_2')^2\lesssim\alpha_1^2$. It follows

$$
\int\langle x\rangle^{-2}|u|^2dxd\omega \lesssim \int(|D_xu|^2+\alpha_1^2|u|^2)dxd\omega.
$$

Changing to coordinates (r,ω) yields, using $dx=\frac{1}{\alpha_1^2}dr$, $D_x=\alpha_1^2D_r$:

$$
\int\langle x(r)\rangle^{-4}|u|^2\frac{1}{\alpha_1^2}drd\omega \lesssim \int(|\alpha_1D_ru|^2+|u|^2)drd\omega.
$$

Putting $v=\frac{1}{\alpha_1}u$ gives

$$
\begin{aligned}
\int\langle x(r)\rangle^{-4}|v|^2drd\omega &\lesssim \int(|\alpha_1D_r\alpha_1v|^2+\alpha_1^2|v|^2)drd\omega,\\
&\lesssim \int(|qD_r\alpha_1v|^2+\alpha_1^2|v|^2)drd\omega
\end{aligned}
$$

which using that $h_0\gtrsim\alpha_1(D_rq^2D_r+1)\alpha_1$ by (G5) completes the proof of the lemma. □

Lemma 8.6. *Let $f,g\in C^\infty(\mathbb{R})$ with $\lim_{|x|\to\infty}f(x)=\lim_{|x|\to\infty}g(x)=0$. Then $f(x)g(h_+)$ and $f(x)g(\tilde h_-)$ are compact.*

Proof. We only prove the lemma for h_+, the proof for $\tilde h_-$ being analogous. We may assume that $f,g\in C_0^\infty(\mathbb{R})$. Let Ω be a bounded domain which contains $\operatorname{supp} f$. Then $f(x)g(h_+)$ sends $L^2(\mathcal{M})$ to $H^2(\Omega)$. But $H^2(\Omega)\hookrightarrow L^2(\Omega)\hookrightarrow L^2(\mathcal{M})$ and the first embedding is compact.

□

8.5.2. *Verification of hypotheses* (A1), (A2). We have already noticed that (G5) implies (A1) (in particular (G5) implies $0 \notin \sigma_{pp}(h_0)$). Let us check (A2). We first check that $h_0^{1/2} k h_0^{-1/2} \in \mathcal{B}(\mathcal{H})$. This will follow from

$$kh_0k \lesssim h_0, \text{ on } C_0^\infty(\mathcal{M}). \tag{8.14}$$

Several terms have to be estimated.

$$\begin{aligned}
& i) \quad -k\alpha_1\partial_r\alpha_2^2\partial_r\alpha_1 k \\
&= -\alpha_1\partial_r k^2\alpha_2^2\partial_r\alpha_1 - \alpha_1\partial_r k\alpha_2^2\alpha_1 k' + \alpha_1^2 k'^2\alpha_2^2 + \alpha_1 k'\alpha_2^2 k\partial_r\alpha_1 \\
&\lesssim -\alpha_1\partial_r q^2\partial_r\alpha_1 - \alpha_1\partial_r k\alpha_2^2\alpha_1 k' + \alpha_1^2 k'^2\alpha_2^2 + \alpha_1 k'\alpha_2 2k\partial_r\alpha_1. \\
& ii) \quad -k\partial_r g^{rr}\partial_r k \\
&= -\alpha_1\partial_r\left(\frac{k}{\alpha_1}\right)^2 g^{rr}\partial_r\alpha_1 - \alpha_1\partial_r k g^{rr}\left(\frac{k}{\alpha_1}\right)' + \left(\frac{k}{\alpha_1}\right)' g^{rr}k\partial_r\alpha_1 \\
&\lesssim -\alpha_1\partial_r q^2\partial_r\alpha_1 - \alpha_1\partial_r k\left(\frac{k}{\alpha_1}\right)' g^{rr} + \left(\frac{k}{\alpha_1}\right)' g^{rr}k\partial_r\alpha_1. \\
& iii) \quad kg^r D_r k = \frac{k^2}{\alpha_1}g^r D_r\alpha_1 + \frac{1}{i}kg^r\left(\frac{k}{\alpha_1}\right)'\alpha_1.
\end{aligned}$$

Summarizing and adding the angular terms we find

$$\begin{aligned}
kh_0k \lesssim\; & \alpha_1 D_r q^2 D_r\alpha_1 - \alpha_1\partial_r k\alpha_2^2\alpha_1 k' + \alpha_1^2(k')^2\alpha_2^2 + \alpha_1 k'\alpha_2^2 k\partial_r\alpha_1 - \alpha_1\partial_r k\left(\frac{k}{\alpha_1}\right)' g^{rr} \\
+\; & \left(\frac{k}{\alpha_1}\right)' kg^{rr}\partial_r\alpha_1 + \frac{1}{i}kg^r\left(\frac{k}{\alpha_1}\right)'\alpha_1 + \frac{k^2}{\alpha_1}g^r D_r\alpha_1 + \alpha_1 D_r\frac{k^2}{\alpha_1}\overline{g}^r - \frac{1}{i}\left(\frac{k}{\alpha_1}\right)' k\alpha_1\overline{g}^r \\
+\; & \sum_{ij}\alpha_3 D_i^*\alpha_{ij}k^2 D_j\alpha_3 + k^2\alpha_4^2 + \sum_{ij} D_i^* k^2 g^{ij} D_j + \sum_i g^i k^2 D_i + D_i^* k^2\overline{g}^i + k^2 f \\
+\; & \sum_{ij} D_i^* k\alpha_3^2\alpha_{ij}(D_j k) - \sum_{ij}\alpha_3^2(D_i^* k)\alpha_{ij}kD_j - \sum_{ij}\alpha_3^2(D_i^* k)\alpha_{ij}(D_j k) \\
+\; & \sum_{ij} D_i^* kg^{ij}(D_j k) - \sum_{ij}(D_i^* k)g^{ij}kD_j - \sum_{ij}(D_i^* k)g^{ij}(D_j k) \\
+\; & \sum_i kg^i(D_i k) - \sum_i (D_i^* k)k\overline{g}^i + 2\mathrm{Im} g^r kk'.
\end{aligned}$$

We have

$$\begin{gathered}
k\alpha_2^2 k' \in T^2,\ \alpha_1^2(k')^2\alpha_2^2 \in T^4,\ k'\alpha_2^2 k \in T^2,\ \frac{k}{q}\left(\frac{k}{\alpha_1}\right)' g^{rr} \in T^\delta,\ k\alpha_1 g^r\left(\frac{k}{\alpha_1}\right)' \in T^\delta, \\
\frac{k^2}{\alpha_1}g^r\frac{1}{q} \in T^\delta,\ g^i k^2 \in T^2,\ k\alpha_3^2\alpha_{ij}(D_i k) \in T^4,\ \alpha_3^2(D_i^* k)\alpha_{ij}(D_j k) \in T^6, \\
kg^{ij}(D_j k) \in T^{4+\delta},\ (D_i^* k)g^{ij}(D_j k) \in T^{6+\delta},\ kg^i(D_i k) \in T^4,\ g^r kk' \in T^{2+\delta}.
\end{gathered}$$

This gives (8.14), using (G5), (8.10)-(8.12) and Lemma 8.5. The estimate for

$$\|(k-z)^{-1}\|_{\mathcal{B}(h_0^{-1/2}\mathcal{H})}$$

is exactly the same with the derivatives of k replaced by $\frac{\partial k}{(k-z)^2}$. Here we also use that

$$\begin{aligned} \|(k-z)^{-1}\|_{\mathcal{B}(\mathcal{H})} &\leq \frac{1}{|\mathrm{Im} z|}, \\ \|(k-z)^{-1}\|_{\mathcal{B}(\mathcal{H})} &\leq \frac{1}{|z| - \|k\|_{\mathcal{B}(\mathcal{H})}}, \end{aligned}$$

the second inequality being valid for $|z| \geq (1+\epsilon)\|k\|_{\mathcal{B}(\mathcal{H})}$, $\epsilon > 0$.

8.5.3. *Verification of hypotheses*(TE1)-(TE3). (TE1) is obvious, let us check (TE2). We check (TE2) for h_+, the proof for $\tilde{h}_-$ being analogous. First note that (G5) for h_+ implies $0 \notin \sigma_{pp}(h_+)$. We have $k_+^2 \in T^2$. This implies the estimate

$$\|k_+ u\| \lesssim \|h_+^{1/2} u\|.$$

We now check that (TE3) is fulfilled. Recall that $w = q^{-1}$.

- (TE3)*a)* follows from (G3).
- (TE3)*b)* is clear.
- (TE3)*c)*: we have already shown in Sect. 8.4.2 that $h_+, \tilde{h}_-$ fulfill (ME2). Let us check (ME1).
  - (ME1)*a)* follows from (G3).
  - (ME1)*b)* is clear.
  - (ME1)*c)*: let us show that $h_+^{-1/2}[h_+, w^{-\epsilon}]w^{\epsilon/2}$ is bounded. We have

$$\begin{aligned} [ih_+, w^{-\epsilon}] &= \alpha_1 D_r \alpha_2^2 \alpha_1 (w^{-\epsilon})' + D_r g^{rr} (w^{-\epsilon})' + g^r (w^{-\epsilon})' + hc \\ &= \alpha_1 D_r \alpha_2^2 \alpha_1 (w^{-\epsilon})' + \alpha_1 D_r \frac{g^{rr}}{\alpha_1} (w^{-\epsilon})' - \frac{1}{i}\alpha_1 \left(\frac{1}{\alpha_1}\right)' g^{rr} (w^{-\epsilon})' + g^r (w^{-\epsilon})' + hc \\ &= \alpha_1 D_r q \alpha + \beta \end{aligned}$$

  with $\alpha \in T^\epsilon$, $\beta \in T^{\epsilon+\delta}$. We have $h_+^{-1/2}\beta w^{\epsilon/2} \in \mathcal{B}(\mathcal{H})$ by Lemma 8.5. We have $h_+^{-1/2}\alpha_1 D_r q \alpha w^{\epsilon/2} \in \mathcal{B}(\mathcal{H})$ by (8.10). The proof for $\tilde{h}_-$ is analogous.
  - (ME1)*d)* follows from Lemma 8.5.
  - (ME1)*e)* follows from Lemma 8.6.
- (TE3)*d)*: the proof is exactly the same as for (A2), we omit the details.
- (TE3)*e)*.: we start with $w[h, i_+]wh_+^{-1/2}$. We have

$$w[ih, i_+]w = w(\alpha_1 D_r \alpha_2^2 \alpha_1 i_+' + D_r g^{rr} i_+' + g^r i_+' + hc)w = \alpha q D_r \alpha_1 + \beta$$

  with $\alpha \in T^\infty$, $\beta \in T^\infty$. This gives $w[h, i_+]wh_+^{-1/2} \in \mathcal{B}(\mathcal{H})$. The proof for the other operators is the same, except for $h_0^{-1/2}[w^{-1}, h_0]w^{1/2}$ for which it is analogous to the proof for $h_+^{-1/2}[h, w^{-\epsilon}]w^{\epsilon/2}$. We omit the details.
- (TE3)*f)* follows from Hardy's inequality, Lemma 8.5.

8.5.4. *Verification of hypotheses* (PE), (B).

- (PE): thanks to (8.10)-(8.12) we see that $\|h_0^{1/2}u\|^2$ is equivalent to

$$\|D_x u\|^2 + \sum_{j=1}^{d-1} \|q(r(x)) D_j u\|^2 + \|q(r(x))u\|^2.$$

As $\psi\left(\frac{x}{n}\right)u \to u$ in $L^2(\mathbb{R}\times S_\omega^{d-1})$ we only have to show that

$$\left[iD_x, \psi\left(\frac{x}{n}\right)\right]u \to 0$$

for $u \in h_0^{-1/2}\mathcal{H}$. We have

$$\left[iD_x, \psi\left(\frac{x}{n}\right)\right]u = \frac{x}{n}\psi'\left(\frac{x}{n}\right)\frac{1}{x}h_0^{-1/2}h_0^{1/2}u.$$

By Lemma 8.5 it is sufficient to show that

$$\frac{x}{n}\psi'\left(\frac{x}{n}\right)v \to 0 \quad \text{for} \quad v \in L^2(\mathbb{R}\times S_\omega^{d-1})$$

which is obvious.

– (B): first note that $w^{-\epsilon}$ clearly sends $D(h_0)$ into itself. We show that (B) is fulfilled. We compute

$$[ih_{0,s}, k] = \alpha_1 D_r \alpha_2^2 k' \alpha_1 + \sum_{ij} D_i^* \alpha_{ij}(\partial_j k)\alpha_3^2 + hc =: C_r + C_\omega.$$

We have

$$\begin{aligned} w^\epsilon C_r w^\epsilon &= \alpha_1 w^\epsilon D_r \alpha_2^2 k' \alpha_1 w^\epsilon + hc \\ &= \alpha_1 D_r \alpha_2^2 k' \alpha_1 w^{2\epsilon} k' \alpha_1 - \frac{1}{i}(w^\epsilon)' \alpha_2^2 \alpha_1^2 w^\epsilon + hc. \end{aligned}$$

Using that

$$\alpha_2^2 w^{2\epsilon} k' \in T^{2-2\epsilon},\ (w^\epsilon)'\alpha_2^2\alpha_1^2 w^\epsilon \in T^{2-2\epsilon},\ \alpha_3^2 \alpha_{ij}(\partial_j k) w^{2\epsilon} \in T^{4-2\epsilon}$$

we find for $\epsilon < 1$

$$w^\epsilon C_r w^\epsilon \lesssim h_0,\ w^\epsilon C_\omega w^\epsilon \lesssim h_0,$$

by Lemma 8.5. We now compute :

$$\begin{aligned} w^\epsilon[i(h_0 - h_{0,s}), k]w^\epsilon &= \sum_{ij} D_i^* q \frac{g^{ij}(iD_j k)w^{2\epsilon}}{q} + \sum_i g^i (D_i k) w^{2\epsilon} \\ &+ \alpha_1 D_r w^{2\epsilon} \alpha_1^{-1} g^{rr} k' - \frac{1}{i}\alpha_1 \left(\frac{w^\epsilon}{\alpha_1}\right)' g^{rr} k' w^\epsilon + hc. \end{aligned}$$

Noting that

$$\begin{aligned} \frac{g^{ij}}{q}(D_j k) w^{2\epsilon} &\in T^{3+\delta-2\epsilon}, \quad g^i (iD_j k) w^{2\epsilon} \in T^{3+\delta-2\epsilon}, \\ g^r k' w^{2\epsilon} &\in T^{2+\delta-2\epsilon}, \quad \frac{w^{2\epsilon} g^{rr} k'}{q\alpha_1} \in T^{2+\delta-2\epsilon}, \\ \left(\frac{w^\epsilon}{\alpha_1}\right)' \alpha_1 g^{rr} k' w^\epsilon &\in T^{2+\delta-2\epsilon}, \end{aligned}$$

we find for $\epsilon > 0$ sufficiently small using (8.10)-(8.12) and Lemma 8.5 that

$$w^\epsilon[i(h_0 - h_{0,s}), k]w^\epsilon \lesssim h_0.$$

Thus (B) is fulfilled.

9. Asymptotic completeness 2 : geometric setting

In this section we will compare the full dynamics to asymptotic spherically symmetric dynamics. We put

$$\begin{aligned} h_{+\infty} &:= h_{0,s}, \quad h_{-\infty} := h_{+\infty} - \ell^2, \\ k_{+\infty} &:= 0, \quad k_{-\infty} := \ell. \end{aligned}$$

The operators

$$\dot{H}_{+\infty} := \begin{pmatrix} 0 & \mathbb{1} \\ h_{+\infty} & 0 \end{pmatrix}, \ \dot{H}_{-\infty} := \begin{pmatrix} 0 & \mathbb{1} \\ h_{-\infty} & 2\ell \end{pmatrix}$$

are selfadjoint on

$$\dot{\mathcal{E}}_{+\infty} = (h_{+\infty})^{-1/2}\mathcal{H} \oplus \mathcal{H} \quad \text{resp. } \dot{\mathcal{E}}_{-\infty} = \Phi(\ell)((h_{+\infty})^{-1/2}\mathcal{H} \oplus \mathcal{H})$$

with domains

$$D(\dot{H}_{+\infty}) = (h_{+\infty})^{-1/2}\mathcal{H} \cap (h_{+\infty})^{-1}\mathcal{H} \oplus \langle h_{+\infty}\rangle^{-1/2}\mathcal{H}, \ D(\dot{H}_{-\infty}) = \Phi(\ell)D(\dot{H}_{+\infty}).$$

Remark 9.1. *We have $\sigma_{pp}(\dot{H}_{\pm\infty}) = \emptyset$. This follows from* [19, Lemme 4.2.1].

Lemma 9.2. *Assume* (G1)-(G7). *Then we have $\dot{\mathcal{E}}_{+\infty} = \dot{\mathcal{E}}_+, \dot{\mathcal{E}}_{-\infty} = \dot{\mathcal{E}}_-$ with equivalent norms.*

Proof. We have to show

$$h_{+\infty} \lesssim h_+ \lesssim h_{+\infty}, \ h_{+\infty} \lesssim \tilde{h}_- \lesssim h_{+\infty}. \tag{9.1}$$

Recalling that $h_{0,s} = h_{+\infty}$, it is sufficient to show (9.1) for $h_{+\infty}$ replaced by $h_{0,s}$. First note that

$$h_{0,s} \lesssim \alpha_1(D_r q^2 D_r + P + 1)\alpha_1. \tag{9.2}$$

(G8) then gives

$$h_{0,s} \lesssim h_+, \quad h_{0,s} \lesssim \tilde{h}_-.$$

By (G5), (G6), the Hardy inequality, Lemma 8.5, and the estimates (8.10)-(8.12) we have

$$h_0 \lesssim h_{0,s}.$$

Now,

$$\begin{aligned} h_+ = h_0 - k_+^2 \lesssim h_0 &\lesssim h_{0,s}, \\ \tilde{h}_- = h_0 - (k_- - \ell)^2 \lesssim h_0 &\lesssim h_{0,s}, \end{aligned}$$

which finishes the proof. □

Lemma 9.3. *Assume* (G1)-(G7). *We have for $\chi \in C_0^\infty(\mathbb{R})$*

$$\chi(\dot{H}_{\pm\infty}) - \chi(\dot{H}_\pm) \in B_\infty(\dot{\mathcal{E}}_\pm).$$

Proof. Let $\chi \in C_0^\infty(\mathbb{R})$. We prove the lemma for $\chi(\dot{H}_{+\infty}) - \chi(\dot{H}_+)$, the proof for $\chi(\dot{H}_{-\infty}) - \chi(\dot{H}_-)$ being analogous. Let us introduce for a positive selfadjoint operator $\hat{h}$ the transformation

$$\mathcal{U}(h) := \frac{1}{\sqrt{2}}\begin{pmatrix} h^{1/2} & i \\ h^{1/2} & -i \end{pmatrix}, \quad \mathcal{U}^{-1}(h) = \frac{1}{\sqrt{2}}\begin{pmatrix} h^{-1/2} & h^{-1/2} \\ -i & i \end{pmatrix}.$$

Note that

$$\mathcal{U}(h_+) : \dot{\mathcal{E}}_\pm \to \mathcal{H} \oplus \mathcal{H},\ \mathcal{U}(h_{+\infty}) : \dot{\mathcal{E}}_{+\infty} \to \mathcal{H} \oplus \mathcal{H}$$

are unitary. We set

$$L_{+\infty} := \mathcal{U}(h_{+\infty})\dot{H}_{+\infty}\mathcal{U}^*(h_{+\infty}),\ L_+ := \mathcal{U}(h_+)\dot{H}_+\mathcal{U}^*(h_+).$$

By [19, Lemmas 6.1.3, A.4.4] we have

$$(1\!\!1 - \mathcal{U}(h_{+\infty})\mathcal{U}^*(h_+))\chi(L_+),\ \chi(L_{+\infty}) - \chi(L_+) \in \mathcal{B}_\infty(\mathcal{H} \oplus \mathcal{H}). \tag{9.3}$$

We now write

$$\begin{aligned}\chi(\dot{H}_{+\infty}) - \chi(\dot{H}_+) &= \mathcal{U}^*(h_{+\infty})(1\!\!1 - \mathcal{U}(h_{+\infty})\mathcal{U}^*(h_+))\chi(L_+)\mathcal{U}(h_+)\\ &\quad + \mathcal{U}^*(h_{+\infty})(\chi(L_{+\infty}) - \chi(L_+))\mathcal{U}(h_{+\infty})\\ &\quad + \mathcal{U}^*(h_{+\infty})\chi(L_+)(1\!\!1 - \mathcal{U}(h_+)\mathcal{U}^*(h_{+\infty}))\mathcal{U}(h_{+\infty}),\end{aligned}$$

which is compact by (9.3). □

Let

$$\dot{R}_{\pm\infty}(z) := (\dot{H}_{\pm\infty} - z)^{-1}.$$

In the same way as for $\dot{H}_\pm$ we can show:

Proposition 9.4. *Assume* (G1)-(G7). *Let* $\epsilon > 0$. *There exists a discrete and closed set* $\mathcal{T}_{\pm\infty} \subset \mathbb{R}$ *such that for all* $\chi \in C_0^\infty(\mathbb{R} \setminus \mathcal{T}_{\pm\infty})$ *and all* $k \in \mathbb{N}$ *we have*

$$\sup_{\epsilon>0,\, \lambda\in\mathbb{R}} \|w^{-\epsilon}\chi(\lambda)\dot{R}^k_{\pm\infty}(\lambda \pm i\epsilon)w^{-\epsilon}\|_{\mathcal{B}(\dot{\mathcal{E}}_{\pm\infty})} < \infty. \tag{9.4}$$

Let $\hat{\mathcal{T}} := \mathcal{S} \cup \mathcal{T}_{\pm\infty}$. The admissible energy cut-offs for $\dot{H}_{\pm\infty}$ are now defined in exactly the same manner as for $\dot{H}$ replacing $\mathcal{S}$ by $\hat{\mathcal{T}}$ in the definition. Let $\mathcal{C}^{H_{\pm\infty}}$ the set of all admissible cut-off functions for $\dot{H}_{\pm\infty}$. We define

$$\begin{aligned}\dot{\mathcal{E}}_{scatt,\pm\infty} &= \{\chi(\dot{H}_{\pm\infty})u;\ \chi \in \mathcal{C}^{H_{\pm\infty}},\ u \in \dot{\mathcal{E}}_{\pm\infty}\},\\ \dot{\mathcal{E}}_{scatt} &= \{\chi(\dot{H})u;\ \chi \in \mathcal{C}^{H_{\pm\infty}},\ u \in \dot{\mathcal{E}}\}.\end{aligned}$$

Theorem 9.5. *Assume* (G1)-(G7).

(i) For all $\varphi^\pm \in \dot{\mathcal{E}}_{scatt,\pm\infty}$ *there exist* $\psi^\pm \in \dot{\mathcal{E}}_{scatt}$ *such that*

$$e^{-it\dot{H}}\psi^\pm - i_\pm e^{-it\dot{H}_{\pm\infty}}\varphi^\pm \to 0,\ t \to \infty \quad \textit{in} \quad \dot{\mathcal{E}}.$$

(ii) For all $\psi^\pm \in \dot{\mathcal{E}}_{scatt}$ *there exist* $\varphi^\pm \in \dot{\mathcal{E}}_{scatt,\pm\infty}$ *such that*

$$e^{-it\dot{H}_{\pm\infty}}\varphi^\pm - i_\pm e^{-it\dot{H}}\psi^\pm \to 0,\ t \to \infty \quad \textit{in} \quad \dot{\mathcal{E}}_{\pm\infty}.$$

Proof. Let $\chi \in \mathcal{C}^{H_{\pm\infty}}$. By Thm. 7.5 it is sufficient to show the existence of the wave operators

$$\begin{aligned}W^\pm_\chi &= \text{s-}\lim_{t\to\infty} e^{it\dot{H}_\pm}e^{-it\dot{H}_{\pm\infty}}\chi(\dot{H}_{\pm\infty}) \quad \text{in} \quad \dot{\mathcal{E}}_\pm,\\ \Omega^\pm_\chi &= \text{s-}\lim_{t\to\infty} e^{it\dot{H}_{\pm\infty}}e^{-it\dot{H}_\pm}\chi(\dot{H}_\pm) \quad \text{in} \quad \dot{\mathcal{E}}_\pm\end{aligned}$$

and that

$$\tilde{\chi}(\dot{H}_\pm)W^\pm_\chi = W^\pm_\chi,\ \tilde{\chi}(\dot{H}_{\pm\infty})\Omega^\pm_\chi = \Omega^\pm_\chi. \tag{9.5}$$

for $\tilde{\chi} \in \mathcal{C}^{H_{\pm\infty}}$ with $\tilde{\chi}\chi = \chi$. The existence of W_χ^+, Ω_χ^+ follows directly from [19, Thm. 6.2.2]. For the existence of W_χ^-, Ω_χ^- note that

$$\begin{aligned} \Phi(\ell)\dot{H}_{-\infty}\Phi^{-1}(\ell) &= \dot{H}^\ell_{-\infty} + \ell\mathbb{1}, \\ \Phi(\ell)\dot{H}_{-}\Phi^{-1}(\ell) &= \dot{H}^\ell_{-} + \ell\mathbb{1}, \end{aligned}$$

where

$$\begin{aligned} \dot{H}^\ell_{-\infty} &= \begin{pmatrix} 0 & \mathbb{1} \\ h_{0,s} & 0 \end{pmatrix}, \\ \dot{H}^\ell_{-} &= \begin{pmatrix} 0 & \mathbb{1} \\ h_0 - (k_- - \ell)^2 & 2(k_- - \ell) \end{pmatrix}. \end{aligned}$$

Again the existence of

$$\begin{aligned} \tilde{W}^-_\chi &= \text{s-}\lim_{t\to\infty} e^{it\dot{H}^\ell_-} e^{-it\dot{H}^\ell_{-\infty}} \chi(\dot{H}^\ell_{-\infty}) \\ \tilde{\Omega}^-_\chi &= \text{s-}\lim_{t\to\infty} e^{it\dot{H}^\ell_{-\infty}} e^{-it\dot{H}^\ell_{-}} \chi(\dot{H}^\ell_{-}) \end{aligned}$$

follows from [19, Thm. 6.2.2]. The existence of W_χ^-, Ω_χ^- then follows applying the transformation $\Phi(\ell)$. The identity (9.5) follows from Lemma 9.3. □

Remark 9.6. *i) Note that the results of* [19] *apply here although the situation considered in* [19] *is slightly different. In* [19] *the cylindrical manifold $\mathbb{R} \times S^{d-1}_\omega$ has one asymptotically euclidean end and one asymptotically hyperbolic end whereas we consider here two asymptotically hyperbolic ends. The situation with two asymptotically hyperbolic ends is simpler, in particular no gluing of the two conjugate operators for the ends in the setting of Mourre theory is necessary.*

ii) As $\sigma_{pp}(\dot{H}_{\pm\infty}) = \emptyset$ and $\dot{H}_\pm$, $\dot{H}_{\pm\infty}$ are selfadjoint, the wave operators

$$W^\pm = \text{s-}\lim_{t\to\infty} e^{it\dot{H}_\pm} e^{-it\dot{H}_{\pm\infty}}$$

exist. In a similar way we obtain the existence of the wave operators

$$\Omega_\pm = \text{s-}\lim_{t\to\infty} e^{it\dot{H}_{\pm\infty}} e^{-itH_\pm} \mathbb{1}^{ac}(\dot{H}_\pm),$$

where $\mathbb{1}^{ac}(\dot{H}_\pm)$ is the projection of the absolutely continuous subspace of $\dot{H}_\pm$.

10. The Klein-Gordon equation on the De Sitter Kerr spacetime

In this section we recall the Klein-Gordon equation on the De Sitter Kerr spacetime, which will be our main example of the geometric framework from Sect. 8.

10.1. The De Sitter Kerr metric in Boyer-Lindquist coordinates.

In Boyer-Lindquist coordinates the De Sitter Kerr space-time is described by a smooth 4-dimensional Lorentzian manifold $\mathcal{M}_{BH} = \mathbb{R}_t \times \mathbb{R}_r \times S^2_\omega$, whose space-time metric is given by

$$\begin{aligned} g &= \frac{\Delta_r - a^2\sin^2\theta\Delta_\theta}{\lambda^2\rho^2}dt^2 + \frac{2a\sin^2\theta((r^2+a^2)^2\Delta_\theta - a^2\sin^2\theta\Delta_r)}{\lambda^2\rho^2}dtd\varphi \\ &- \frac{\rho^2}{\Delta_r}dr^2 - \frac{\rho^2}{\Delta_\theta}d\theta^2 - \frac{\sin^2\theta\sigma^2}{\lambda^2\rho^2}d\varphi^2, \\ \rho^2 &= r^2 + a^2\cos^2\theta, \quad \Delta_r = \left(1 - \frac{\Lambda}{3}r^2\right)(r^2+a^2) - 2Mr, \\ \Delta_\theta &= 1 + \frac{1}{3}\Lambda a^2\cos^2\theta,\ \sigma^2 = (r^2+a^2)^2\Delta_\theta - a^2\Delta_r\sin^2\theta,\ \lambda = 1 + \frac{1}{3}\Lambda a^2. \end{aligned} \tag{10.1}$$

Here $\Lambda > 0$ is the cosmological constant, $M > 0$ is the mass of the black hole and a its angular momentum per unit mass. The metric is defined for $\Delta_r > 0$, we assume that this is fulfilled on an open interval $]r_-, r_+[$. (For $a = 0$, this is true when $9\Lambda M^2 < 1$; it remains true if we take a small enough).

Note that the vector fields ∂_t and ∂_φ are Killing. The De Sitter-Schwarzschild metric ($a = 0$) is a special case of the above. The set $\{\rho^2 = 0\}$ is a true curvature singularity. In contrast to ρ^2 the roots of Δ_r are mere coordinate singularities. r_- and r_+ represent *event horizons* and we will only be interested in the region $r_- < r < r_+$. This region is not stationary in the sense that there exists no global time-like Killing vector field. In particular there are regions in $\mathbb{R}_t \times]r_-, r_+[_r \times S^2_\omega$ in which ∂_t becomes space-like. Indeed, the function Δ_r' has a single zero r_{max} on $]r_-, r_+[$. On $]r_-, r_{max}[$ Δ_r is strictly increasing, on $]r_{max}, r_+[$ Δ_r is strictly decreasing. Therefore there exist $r_1(\theta)$, $r_2(\theta)$ defined on $]0, \pi[$ such that

$$\begin{aligned} \Delta_r - a^2\sin^2\theta\Delta_\theta < 0 &\quad \text{on }]r_-, r_1(\theta)[, \\ \Delta_r - a^2\sin^2\theta\Delta_\theta > 0 &\quad \text{on }]r_1(\theta), r_2(\theta)[, \\ \Delta_r - a^2\sin^2\theta\Delta_\theta < 0 &\quad \text{on }]r_2(\theta), r_+[. \end{aligned}$$

As a consequence the vector field ∂_t is

– time-like on $\{(t, r, \theta, \varphi) : r_1(\theta) < r < r_2(\theta)\}$,

– spacelike on $\{(t, r, \theta, \varphi) : r_- < r < r_1(\theta)\} \cup \{(t, r, \theta, \varphi : r_2(\theta) < r < r_+\} =: \mathcal{A}_- \cup \mathcal{A}_+$.

The regions $\mathcal{A}_\pm$ are called *ergo-spheres*. Of particular interest are the *locally non rotating observers*. These observers have four velocity

$$u^a = \frac{\nabla^a t}{(\nabla_b t \nabla^b t)^{1/2}}.$$

They rotate with coordinate angular velocity

$$\Omega = -\frac{g_{t\varphi}}{g_{\varphi\varphi}} = \frac{a((r^2+a^2)\Delta_\theta - \Delta_r a^2\sin^2\theta)}{\sigma^2}. \tag{10.2}$$

Note that this angular velocity has finite limits at both horizons:

$$\Omega_\pm := \Omega(r_\pm, \theta) = \frac{a}{r_\pm^2 + a^2}. \tag{10.3}$$

10.2. **The Klein-Gordon equation on the De Sitter Kerr space-time.** We now reduce the Klein-Gordon equation on the De Sitter Kerr space time to the abstract form (1.2).

A standard computation using $\Box_g = |g|^{-\frac{1}{2}}\partial_a |g|^{\frac{1}{2}} g^{ab}\partial_b$ yields for the De Sitter Kerr metric:

$$\begin{aligned}&\Big(\frac{\sigma^2\lambda^2}{\rho^2\Delta_\theta\Delta_r}\partial_t^2 - 2\frac{a(\Delta_r-(r^2+a^2)\Delta_\theta)\lambda^2}{\rho^2\Delta_\theta\Delta_r}\partial_\varphi\partial_t\\ &- \frac{(\Delta_r - a^2\sin^2\theta\Delta_\theta)\lambda^2}{\rho^2\Delta_\theta\Delta_r\sin^2\theta}\partial_\varphi^2 - \frac{1}{\rho^2}\partial_r\Delta_r\partial_r - \frac{1}{\sin\theta\rho^2}\partial_\theta\sin\theta\Delta_\theta\partial_\theta + m^2\Big)\psi = 0.\end{aligned} \tag{10.4}$$

We multiply to the left the equation by $c^2 = \frac{\rho^2\Delta_r\Delta_\theta}{\lambda^2\sigma^2}$ and obtain:

$$\begin{aligned}&\Big(\partial_t^2 - 2\frac{a(\Delta_r-(r^2+a^2)\Delta_\theta)}{\sigma^2}\partial_\varphi\partial_t\\ &- \frac{(\Delta_r - a^2\sin^2\theta\Delta_\theta)}{\sin^2\theta\sigma^2}\partial_\varphi^2 - \frac{\Delta_r\Delta_\theta}{\lambda^2\sigma^2}\partial_r\Delta_r\partial_r - \frac{\Delta_r\Delta_\theta}{\lambda^2\sin\theta\sigma^2}\partial_\theta\sin\theta\Delta_\theta\partial_\theta + \frac{\rho^2\Delta_r\Delta_\theta}{\lambda^2\sigma^2}m^2\Big)\psi = 0.\end{aligned} \tag{10.5}$$

We now consider the unitary transform

$$U: \begin{array}{rcl} L^2(\mathcal{M}; \frac{\sigma^2}{\Delta_r\Delta_\theta}drd\omega) & \to & L^2(\mathcal{M}; drd\omega)\\ \psi & \mapsto & \frac{\sigma}{\sqrt{\Delta_r\Delta_\theta}}\psi\end{array}$$

If ψ solves (10.5), then $u = U\psi$ solves

$$\begin{aligned}&\Big(\partial_t^2 - 2\frac{a(\Delta_r-(r^2+a^2)\Delta_\theta)}{\sigma^2}\partial_\varphi\partial_t - \frac{(\Delta_r - a^2\sin^2\theta\Delta_\theta)}{\sin^2\theta\sigma^2}\partial_\varphi^2 - \frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma}\partial_r\Delta_r\partial_r\frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma}\\ &- \frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sin\theta\sigma}\partial_\theta\sin\theta\Delta_\theta\partial_\theta\frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma} + \frac{\rho^2\Delta_r\Delta_\theta}{\lambda^2\sigma^2}m^2\Big)u = 0.\end{aligned} \tag{10.6}$$

We introduce a Regge-Wheeler type coordinate x by the requirement :

$$\frac{dx}{dr} = \lambda\frac{r^2+a^2}{\Delta_r}.$$

We then introduce the unitary transform

$$\mathcal{V}: \begin{array}{rcl} L^2(]r_-, r_+[_r\times S^2) & \to & L^2(\mathbb{R}\times S^2, dxd\omega),\\ v(r,\omega) & \mapsto & \sqrt{\frac{\Delta_r}{\lambda(r^2+a^2)}}v(r(x),\omega).\end{array}$$

Let u be a solution of the Klein-Gordon equation (10.6) and $\psi = \sqrt{\frac{\Delta_r}{\lambda(r^2+a^2)}}u$. Then ψ fulfills:

$$\begin{aligned}&\Big(\partial_t^2 - 2\frac{a(\Delta_r-(r^2+a^2)\Delta_\theta)}{\sigma^2}\partial_\varphi\partial_t - \frac{(\Delta_r - a^2\sin^2\theta\Delta_\theta)}{\sin^2\theta\sigma^2}\partial_\varphi^2\\ &- \frac{\sqrt{(r^2+a^2)\Delta_\theta}}{\sigma}\partial_x(r^2+a^2)\partial_x\frac{\sqrt{(r^2+a^2)\Delta_\theta}}{\sigma} - \frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sin\theta\sigma}\partial_\theta\sin\theta\Delta_\theta\partial_\theta\frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma} + \frac{\rho^2\Delta_r\Delta_\theta}{\lambda^2\sigma^2}m^2\Big)\psi = 0.\end{aligned} \tag{10.7}$$

11. Asymptotic completeness 3 : The De Sitter Kerr case

In this section we state the main theorems for the De Sitter Kerr spacetime. The proofs are given in Sect. 12.

We consider the Klein-Gordon equation (10.7) and write it in the usual form

$$(\partial_t^2 - 2ik\partial_t + h)\psi = 0.$$

Let

$$\mathcal{H}^n = \{u \in L^2(\mathbb{R} \times S^2) : \ (D_\varphi - n)u = 0\}, \ n \in \mathbb{Z}. \tag{11.1}$$

We construct the energy spaces $\dot{\mathcal{E}}^n$, $\mathcal{E}^n$ as well as the Klein-Gordon operators H^n, $\dot{H}^n$ as in Sect. 2. Also let $i_\pm \in C^\infty(\mathbb{R})$, $i_- = 0$ in a neighborhood of ∞, $i_+ = 0$ in a neighborhood of $-\infty$ and $i_-^2 + i_+^2 = 1$. We will use two types of comparison dynamics :

– a separable comparison dynamics,
– asymptotic profiles.

11.1. Uniform boundedness of the evolution.

Theorem 11.1. *There exists $a_0 > 0$ such that for $|a| < a_0$ the following holds: for all $n \in \mathbb{Z}$, there exists $C_n > 0$ such that*

$$\|e^{-it\dot{H}^n}u\|_{\dot{\mathcal{E}}^n} \le C_n\|u\|_{\dot{\mathcal{E}}^n}, \ u \in \dot{\mathcal{E}}^n, \ t \in \mathbb{R}. \tag{11.2}$$

Note that for $n = 0$ the Hamiltonian $\dot{H}^n = \dot{H}^0$ is selfadjoint, therefore the only issue is $n \neq 0$.

Because of the existence of a zero resonance the evolution is not expected to be uniformly bounded on the inhomogeneous energy space. This is already the case in the De Sitter Schwarzschild case, ie if $a = 0$. In fact from [5, Thm. 1.3], denoting by r the zero resonance state, we have for $\chi \in C_0^\infty(\mathbb{R})$:

$$\chi e^{-itH}\chi u \ = \ \gamma\begin{pmatrix} r\chi(\chi r|u_1) \\ 0 \end{pmatrix} + E(t), \quad \gamma > 0 \quad \text{and} \tag{11.3}$$

$$\|E(t)\|_{\mathcal{E}} \ \lesssim \ e^{-\epsilon t}\|\langle -\Delta_{S^2}\rangle u\|_{\mathcal{E}}, \tag{11.4}$$

with $\epsilon > 0$. Note that in [5] the norm

$$\|u_1\|^2 + (h_0u|u) + \int_0^1\int_{S^2}|u_0|^2(x,\omega)dxd\omega$$

is used, but that the same proof also gives the estimate (11.4). Now suppose that e^{-itH} is uniformly bounded on $\mathcal{E}$. Then we obtain from (11.3) in the limit $t \to \infty$:

$$\|r\chi(r\chi|u_1)\|_{\mathcal{E}} \lesssim \|u\|_{\mathcal{E}}, \quad u \in C_0^\infty(\mathbb{R} \times S^2) \oplus C_0^\infty(\mathbb{R} \times S^2)$$

and thus

$$\|r\chi(r\chi|u_1)\|_{\mathcal{H}} \lesssim \|u\|_{\mathcal{E}}, \quad u \in \mathcal{E}$$

by density. Here $\mathcal{H} = L^2(\mathbb{R} \times S^2, dxd\omega)$. It follows

$$\|r\chi(r\chi|v)\|_{\mathcal{H}} \lesssim \|v\|_{\mathcal{H}}, \quad v \in \mathcal{H}.$$

Thus $\|r\chi\|_{\mathcal{H}} \lesssim 1$ uniformly in χ which implies $r \in \mathcal{H}$ which is false. Therefore the evolution is not uniformly bounded on $\mathcal{E}$, neither on $\mathcal{E}^0$. It is however bounded on $\mathcal{E}^n$ for all $n \neq 0$.

11.2. **Separable comparison dynamics.** Let $\ell_\pm = \Omega_\pm n$. We put:

$$h_{\pm\infty} := -\ell_\pm^2 - \partial_x^2 + \frac{\Delta_r}{\lambda^2(r^2+a^2)}P + \Delta_r m^2, \ k_{\pm\infty} := \ell_\pm,$$

where

$$P := -\frac{\lambda^2}{\sin^2\theta}\partial_\varphi^2 - \frac{1}{\sin\theta}\partial_\theta \sin\theta\Delta_\theta\partial_\theta.$$

In the case $n = 0$ P might have a zero eigenvalue and the natural energy spaces associated to h_0 and $h_{\pm\infty}$ may be different in the massless case. We will therefore consider the case $n = 0$ only in the massive case. Let $\dot{\mathcal{E}}^n_{\pm\infty}$, $\dot{H}^n_{\pm\infty}$ be the homogeneous energy spaces and operators associated to $(h_{\pm\infty}, k_{\pm\infty})$ according to Sect. 2.

Theorem 11.2. *There exists $a_0 > 0$ such that for $|a| < a_0$ and $n \in \mathbb{Z}\setminus\{0\}$ the following holds:*

- *The wave operators*

$$W^\pm = \text{s-}\lim_{t\to\infty} e^{it\dot{H}^n} i_\pm e^{-it\dot{H}^n_{\pm\infty}} \tag{11.5}$$

 exist as bounded operators $W^\pm \in \mathcal{B}(\dot{\mathcal{E}}^n_{\pm\infty}; \dot{\mathcal{E}}^n)$.
- *The inverse wave operators*

$$\Omega^\pm = \text{s-}\lim_{t\to\infty} e^{it\dot{H}^n_{\pm\infty}} i_\pm e^{-it\dot{H}^n} \tag{11.6}$$

 exist as bounded operators $\Omega^\pm \in \mathcal{B}(\dot{\mathcal{E}}^n; \dot{\mathcal{E}}^n_{\pm\infty})$.

(11.5) *and* (11.6) *also hold for $n = 0$ if $m > 0$.*

11.3. **Asymptotic profiles.** We now introduce the Hamiltonians $\dot{H}_r$, $\dot{H}_l$ which describe the simplest possible asymptotic comparison dynamics. Let

$$h^n_{r/l} = -\partial_x^2 - \ell^2_{+/-}, \ k_{r/l} = \ell_{+/-},$$

acting on $\mathcal{H}^n$ defined in (11.1).

We associate to these operators the natural homogeneous energy spaces $\dot{\mathcal{E}}^n_{l/r}$ and Hamiltonians $\dot{H}^n_{l/r}$.Note that the solution of

$$\left\{\begin{array}{rcl} (\partial_t^2 - 2i\ell_\pm\partial_t - \partial_x^2 - \ell_\pm^2)u &=& 0,\\ u|_{t=0} &=& u_0,\\ \partial_t u|_{t=0} &=& u_1 \end{array}\right. \tag{11.7}$$

can be computed explicitly. Indeed if u is the solution of (11.7), then $v = e^{-i\ell_\pm t}u$ fulfills

$$\left\{\begin{array}{rcl} (\partial_t^2 - \partial_x^2)v &=& 0,\\ v|_{t=0} &=& u_0,\\ \partial_t v|_{t=0} &=& u_1 - i\ell_\pm u_0 \end{array}\right. \tag{11.8}$$

Thus for smooth data the explicit solution of (11.7) is given by

$$u_0(t,x,\omega) = \frac{e^{i\ell_\pm t}}{2}\left(u_0(x+t,\omega) + u_0(x-t,\omega) + \int_{x-t}^{x+t}(u_1(\tau,\omega) - i\ell_\pm u_0(\tau,\omega))d\tau\right).$$

Let us denote the cutoffs $i_{+/-}$ by $i_{r/l}$.

The spaces $i_l\dot{\mathcal{E}}_l^n$ and $i_r\dot{\mathcal{E}}_r^n$ are not included in $\dot{\mathcal{E}}^n$ and the group $e^{-it\dot{H}^n_{r/l}}$ doesn't improve regularity. There is therefore no chance that the limits

$$W^+u = \lim_{t\to\infty} e^{it\dot{H}^n} i_{r/l} e^{-it\dot{H}^n_{r/l}} u$$

exist for all $u \in \dot{\mathcal{E}}^n_{r/l}$. We will first show the existence of the limits on smaller spaces and then extend the wave operators by continuity. Let $\{\lambda_q : q \in \mathbb{N}\} = \sigma(P)$ and $Z_q = \mathbb{1}_{\{\lambda_q\}}(P)\mathcal{H}$. Then

$$D(h_0) = D(h_{0,s}) = \{u \in \mathcal{H} : \sum_{q\in\mathbb{N}} \|h_0^{s,q}\mathbb{1}_{\{\lambda_q\}}(P)u\|^2 < \infty\},$$

where $h_0^{s,q}$ is the restriction of $h_{0,s}$ to $L^2(\mathbb{R}) \otimes Z_q$. Let

$$\begin{aligned} W_q &:= (L^2(\mathbb{R})\otimes Z_q)\oplus(L^2(\mathbb{R})\otimes Z_q),\ \mathcal{E}^{q,n}_{l/r} := \mathcal{E}^n_{r/l}\cap W_q,\\ \mathcal{E}^{fin,n}_{l/r} &:= \left\{u\in\mathcal{E}^n_{l/r} : \exists Q>0,\, u\in\oplus_{q\le Q}\mathcal{E}^{q,n}_{l/r}\right\}. \end{aligned}$$

Theorem 11.3. *There exists $a_0 > 0$ such that for all $|a| < a_0$ and $n \in \mathbb{Z}\setminus\{0\}$ the following holds:*

i) For all $u \in \mathcal{E}^{fin,n}_{r/l}$ the limits

$$W_{r/l}u = \lim_{t\to\infty} e^{it\dot{H}^n} i^2_{r/l} e^{-it\dot{H}^n_{r/l}} u$$

exist in $\dot{\mathcal{E}}^n$. The operators $W_{r/l}$ extend to bounded operators $W_{r/l} \in \mathcal{B}(\dot{\mathcal{E}}^n_{r/l}; \dot{\mathcal{E}}^n)$.

ii) The inverse wave operators

$$\Omega_{r/l} = \text{s-}\lim_{t\to\infty} e^{it\dot{H}^n_{r/l}} i^2_{r/l} e^{-it\dot{H}^n}$$

exist in $\mathcal{B}(\dot{\mathcal{E}}^n; \dot{\mathcal{E}}^n_{r/l})$.

i), ii) also hold for $n = 0$ if $m > 0$.

12. Proof of the main theorems for the De Sitter Kerr spacetime

We want to apply the geometric setting developed in Sect. 8. To do so, we have to reduce ourselves to $\ell_+ = 0$ by a change of coordinates. We introduce the new coordinate

$$\tilde{\varphi} = \varphi - \frac{a}{r_+^2 + a^2}t,$$

the other coordinates remain unchanged. We will denote $\tilde{\varphi}$ again by φ in the following. In the new coordinates (10.7) writes :

$$\begin{aligned} &\left(\left(\partial_t - \frac{a}{r_+^2+a^2}\partial_\varphi\right)^2 - 2\frac{a(\Delta_r-(r^2+a^2)\Delta_\theta)}{\sigma^2}\partial_\varphi\left(\partial_t - \frac{a}{r_+^2+a^2}\partial_\varphi\right) - \frac{(\Delta_r - a^2\sin^2\theta\Delta_\theta)}{\sin^2\theta\sigma^2}\partial_\varphi^2\right.\\ (12.1)\quad &\left. - \frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma}\partial_r\Delta_r\partial_r\frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma} - \frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sin\theta\sigma}\partial_\theta\sin\theta\Delta_\theta\partial_\theta\frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma} + \frac{\rho^2\Delta_r\Delta_\theta}{\lambda^2\sigma^2}m^2\right)\psi = 0, \end{aligned}$$

i.e.

$$\begin{aligned}
&\left(\partial_t^2 - 2\left(\frac{a}{(r_+^2+a^2)} + \frac{a(\Delta_r-(r^2+a^2)\Delta_\theta)}{\sigma^2}\right)\partial_\varphi\partial_t\right.\\
&+ \left(\frac{a^2}{(r_+^2+a^2)^2} + 2\frac{a^2(\Delta_r-(r^2+a^2)\Delta_\theta)}{\sigma^2(r_+^2+a^2)} - \frac{(\Delta_r-a^2\sin^2\theta\Delta_\theta)}{\sin^2\theta\sigma^2}\right)\partial_\varphi^2\\
&\left.- \frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma}\partial_r\Delta_r\partial_r\frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma} - \frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sin\theta\sigma}\partial_\theta\sin\theta\Delta_\theta\partial_\theta\frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma} + \frac{\rho^2\Delta_r\Delta_\theta}{\lambda^2\sigma^2}m^2\right)\psi = 0. \qquad (12.2)
\end{aligned}$$

Let us put:

$$\begin{aligned}
k &:= \left(\frac{a}{(r_+^2+a^2)} + \frac{a(\Delta_r-(r^2+a^2)\Delta_\theta)}{\sigma^2}\right)D_\varphi,\\
h &:= \left(\frac{a^2}{(r_+^2+a^2)^2} + 2\frac{a^2(\Delta_r-(r^2+a^2)\Delta_\theta)}{\sigma^2(r_+^2+a^2)} - \frac{(\Delta_r-a^2\sin^2\theta\Delta_\theta)}{\sin^2\theta\sigma^2}\right)\partial_\varphi^2\\
&\quad - \frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma}\partial_r\Delta_r\partial_r\frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma} - \frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sin\theta\sigma}\partial_\theta\sin\theta\Delta_\theta\partial_\theta\frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma} + \frac{\rho^2\Delta_r\Delta_\theta}{\lambda^2\sigma^2}m^2.
\end{aligned}$$

Noting that the coordinate change $\varphi \to \tilde\varphi$ corresponds to the unitary transform $e^{-it\Omega_+D_\varphi}$ and using Subsect. 2.5.3 we see that it is sufficient to show the corresponding theorems of Sect. 11 for the operators h, k. We put $h_0 := h + k^2$. A tedious calculation gives :

$$\begin{aligned}
h_0 &= -\frac{\rho^4\Delta_r\Delta_\theta}{\sigma^4\sin^2\theta}\partial_\varphi^2\\
&\quad - \frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma}\partial_r\Delta_r\partial_r\frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma} - \frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sin\theta\sigma}\partial_\theta\sin\theta\Delta_\theta\partial_\theta\frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma} + \frac{\rho^2\Delta_r\Delta_\theta}{\lambda^2\sigma^2}m^2. \qquad (12.3)
\end{aligned}$$

We put

$$\begin{aligned}
h_0^n &:= \frac{(\rho^4-\sigma^2)\Delta_r\Delta_\theta}{\sigma^4\sin^2\theta}n^2\\
&\quad - \frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma}\partial_r\Delta_r\partial_r\frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma} + \frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma}P\frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma} + \frac{\rho^2\Delta_r\Delta_\theta}{\lambda^2\sigma^2}m^2, \qquad (12.4)\\
k^n &:= \left(\frac{a}{(r_+^2+a^2)} + \frac{a(\Delta_r-(r^2+a^2)\Delta_\theta)}{\sigma^2}\right)n. \qquad (12.5)
\end{aligned}$$

We will drop in the following the index n which is implicit in the operators.

12.1. **Verification of the geometric hypotheses.**

Let us recall that

$$P = -\frac{\lambda^2}{\sin^2\theta\Delta_\theta}\partial_\varphi^2 - \frac{1}{\sin\theta}\partial_\theta\sin\theta\Delta_\theta\partial_\theta.$$

With this choice of P (G1) is clearly fulfilled. We now put

$$h_{0,s} := -\frac{\sqrt{\Delta_r}}{\lambda(r^2+a^2)}\partial_r\Delta_r\partial_r\frac{\sqrt{\Delta_r}}{\lambda(r^2+a^2)} + \frac{\sqrt{\Delta_r}}{\lambda(r^2+a^2)}P\frac{\sqrt{\Delta_r}}{\lambda(r^2+a^2)} + \Delta_r m^2.$$

Recall that $q(r) := \sqrt{(r_+ - r)(r - r_-)}$. We write $\Delta_r = q^2(r)P_2(r)$, where P_2 is a polynomial of degree 2. It is easy to see that (G2) is fulfilled with

$$\alpha_1^\pm = \alpha_3^\pm := \frac{\sqrt{P_2(r_\pm)}}{\lambda(r_\pm^2 + a^2)},\ \alpha_2^\pm := \sqrt{P_2(r_\pm)},\ \alpha_4^\pm := m^2\frac{\sqrt{P_2(r_\pm)}}{(r_\pm^2 + a^2)\lambda^2}.$$

We also put

$$\begin{aligned} k_{s,v} &:= k_s^n := \left(\frac{a}{r_+^2 + a^2} - \frac{a}{r^2 + a^2}\right)n, \\ k_{s,v}^- &:= \frac{an}{(r_+^2 + a^2)(r_-^2 + a^2)}(r_- - r_+)(r_- + r_+), \\ k_{s,r} &= k_{s,r}^- = 0. \end{aligned}$$

With these choices (G3) is clearly fulfilled. (G4) follows from (12.4). Let

$$\begin{aligned} g_1 &:= \frac{\sqrt{\Delta_r \Delta_\theta}}{\lambda\sigma} \in T^1, \quad g_0 := \frac{\sqrt{\Delta_r}}{\lambda(r^2 + a^2)} \in T^1, \\ p_1 &:= \frac{\sqrt{\Delta_r \Delta_\theta}}{\sigma} \in T^1, \quad p_0 := \frac{\sqrt{\Delta_r}}{r^2 + a^2} \in T^1. \end{aligned}$$

We have

$$g_1 - g_0 \in T^3, \quad p_1 - p_0 \in T^3.$$

An elementary calculation gives :

$$\begin{aligned} g^{rr} &= (g_0 - g_1)(g_0 + g_1)\Delta_r \in T^5, \\ g^r &= i((\partial_r g_1)g_1 - (\partial_r g_0)g_0)\Delta_r \in T^3, \\ g^{\theta\theta} &= (p_0 - p_1)(p_0 + p_1) \in T^3, \\ g^\theta &= i((\partial_\theta p_1)p_1 - (\partial_\theta p_0)p_0) \in T^2, \\ f &= ((\partial_r g_0)^2 - (\partial_r g_1)^2)\Delta_r + ((\partial_\theta p_0)^2 - (\partial_\theta p_1)^2) + \frac{m^2\Delta_r\rho^2\Delta_\theta}{\lambda^2\sigma^2} - \Delta_r m^2 \\ &+ \left(\frac{\rho^4\Delta_r\Delta_\theta}{\sigma^4\sin^2\theta} - \frac{\Delta_r}{(r^2 + a^2)^2\sin^2\theta}\right)n^2 \in T^2, \\ g^{\varphi\varphi} &= g^\varphi = 0. \end{aligned}$$

Note that because of the diagonalization w.r.t. D_φ, we can put $g^{\varphi\varphi}$ and g^φ into f. It follows that hypothesis (G6) is fulfilled. Let us now check (G5). We consider the case $n = 0$ only if $m > 0$. Also (G5) will only be satisfied if $|a| < a_1$ for some a_1 independent of n. We first show that (G5) is fulfilled for $h_{0,s}^n$. We have

$$\begin{aligned} h_{0,s}^n &= \alpha_1(D_r\Delta_r D_r + P + (r^2 + a^2)^2 m^2)\alpha_1 \\ &\gtrsim \alpha_1(D_r q^2 D_r + P + 1)\alpha_1, \end{aligned}$$

with $\alpha_1 = \frac{\sqrt{\Delta_r}}{\lambda(r^2+a^2)}$ because $P \gtrsim 1$ for $n \neq 0$ and we suppose $m > 0$ for $n = 0$. Let now

$$\begin{aligned}\tilde{h}_0^n &= \frac{\rho^4\Delta_r\Delta_\theta}{\sigma^2\Delta_\theta\sin^2\theta}n^2 - \frac{\sqrt{\Delta_r}}{\lambda(r^2+a^2)}\partial_r\Delta_r\partial_r\frac{\sqrt{\Delta_r}}{\lambda(r^2+a^2)}\\ &- \frac{\sqrt{\Delta_r}}{\lambda(r^2+a^2)\sin\theta}\partial_\theta\sin\theta\Delta_\theta\partial_\theta\frac{\sqrt{\Delta_r}}{\lambda(r^2+a^2)} + \frac{\rho^2\Delta_r m^2\Delta_\theta}{\lambda^2\sigma^2}\\ &\gtrsim \alpha_1(D_r q^2 D_r + P + 1)\alpha_1.\end{aligned}$$

We then compute

$$\begin{aligned}h_0^n - \tilde{h}_0^n &= -\left(\frac{1}{\sigma} - \frac{1}{(r^2+a^2)\sqrt{\Delta_\theta}}\right)\frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda}\partial_r\Delta_r\partial_r\frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma}\\ &- \frac{\sqrt{\Delta_r}}{\lambda(r^2+a^2)}\partial_r\Delta_r\partial_r\frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda}\left(\frac{1}{\sigma} - \frac{1}{(r^2+a^2)\sqrt{\Delta_\theta}}\right)\\ &- \left(\frac{1}{\sigma} - \frac{1}{(r^2+a^2)\sqrt{\Delta_\theta}}\right)\frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sin\theta}\partial_\theta\sin\theta\Delta_\theta\partial_\theta\frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma}\\ &- \frac{\sqrt{\Delta_r}}{(r^2+a^2)\lambda\sin\theta}\partial_\theta\sin\theta\Delta_\theta\partial_\theta\frac{\sqrt{\Delta_r}}{\lambda}\left(\frac{1}{\sigma} - \frac{1}{(r^2+a^2)\sqrt{\Delta_\theta}}\right).\end{aligned}$$

We compute

$$\left(\frac{1}{\sigma} - \frac{1}{(r^2+a^2)\sqrt{\Delta_\theta}}\right) = \frac{a^2\Delta_r\sin^2\theta}{\sigma^2(r^2+a^2)^2\Delta_\theta}\left(\frac{1}{\sigma} + \frac{1}{(r^2+a^2)\Delta_\theta}\right)^{-1} =: a^2 g_a.$$

We have $g_a \in T^2$ uniformly in a meaning that

$$\forall\,\alpha,\beta \in \mathbb{N}, \quad |\partial_r^\alpha\partial_\omega^\beta f_a| \leq C_{\alpha\beta}q(r)^{2-2\alpha}$$

with $C_{\alpha,\beta}$ independent of a. We then compute

$$\begin{aligned}-a^2 g_a\frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda}\partial_r\Delta_r\partial_r\frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma} &= -a^2\frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma}\partial_r\tilde{g}_a\Delta_r\partial_r\frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma}\\ &+ a^2\tilde{g}_a'\frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma}\Delta_r\partial_r\frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma}\\ &\gtrsim -a^2 h_0^n,\end{aligned}$$

where $\tilde{g}_a = g_a\sigma \in T^2$ uniformly in a. Using similar arguments for the other terms we find

$$h_0^n - \tilde{h}_0^n \gtrsim -a^2 h_0^n - a^2\tilde{h}_0^n$$

and thus for a small enough (independently of n):

$$h_0^n \gtrsim \tilde{h}_0^n \gtrsim \alpha_1(D_r q^2 D_r + P + 1)\alpha_1. \tag{12.6}$$

This is (G5) for h_0^n. We also have

$$k_{p,v} = \left(\frac{a\Delta_r}{\sigma^2} + \frac{a^3\sin^2\theta\Delta_r}{(r^2+a^2)\sigma^2}\right)n \in T^2, \quad k_{p,r} = 0.$$

This implies (G7). Let us now check (G8). Recall that $\ell = k_{s,v}^-$. We construct h_+, $\tilde{h}_-$ and $k_\pm$ as in Subsect. 8.3.1. We have

$$h_+ = h_0 - k_+^2,\ k_+^2 \leq C_+ a^2 n^2(r_+ - r)(r - r_-).$$

Observing that $P \geq \frac{n^2}{\sin^2\theta}$ we obtain using (12.6):

$$h_0^n \gtrsim \alpha_1(D_r q^2 D_r + P + 1)\alpha_1 + n^2(r_+ - r)(r - r_-).$$

Thus fixing a cut-off scale $\epsilon > 0$ (see Section 4.1) there exists $a_2 > 0$ (independent of n) such that for all $|a| < a_2$ and $n \in \mathbb{Z}$ we have :

$$h_+,\ \tilde{h}_- \gtrsim \alpha_1(r)(D_r q^2(r) D_r + \tilde{P} + 1)\alpha_1(r).$$

In particular (G5) is fulfilled for h_+, $\tilde{h}_-$ if a is small enough. Thus (G8) is fulfilled. In the sequel we assume $|a| < a_0$, where a_0 is such that all the geometric hypotheses are fulfilled for all $n \in \mathbb{Z} \setminus \{0\}$ $(m = 0)$ resp. all $n \in \mathbb{Z}$ $(m > 0)$ if $|a| < a_0$.

12.2. **Proof of Thm. 11.1.** Thm. 11.1 will follow from Thm. 6.1, provided we show that the set $\mathcal{S}$ of singular points is empty. We recall that the sets $\mathcal{S}$, $\mathcal{T}$, $\mathcal{T}_\pm$ were defined in Subsect. 5.1 and that we showed in Prop. 5.10 that $\mathcal{S} \subset \mathcal{T} \cup \mathcal{T}_- \cup \mathcal{T}_+$. Therefore the proof of Thm. 11.1 will follow from

Proposition 12.1. *i) There exists $a_1 > 0$ such that for $|a| < a_1$, $n \neq 0$:*

$$\sigma^{\mathbb{C}}_{\rm pp}(\dot{H}) = \mathcal{T} = \emptyset.$$

ii) One has $\mathcal{T}_\pm = \emptyset$ for $n \neq 0$.

Proof. i) essentially follows from the work of Dyatlov [9]. Let us first prove that $\sigma^{\mathbb{C}}(\dot{H}) = \emptyset$. By [9, Thm. 4] we have $\rho(h, k) \cap \{\mathrm{Im} z > 0\} = \emptyset$ for $a > 0$ sufficiently small. Then we apply Prop. 2.15.

Let us now prove that $\mathcal{T} = \emptyset$, i.e. that $r(z) := w^{-\epsilon} p^{-1}(z) w^{-\epsilon}$ has no real poles. We replace the weight $w^{-\epsilon}$ by $\cosh(\epsilon x)^{-1}$ which is equivalent and holomorphic in a neighborhood of the real axis. We will note this new weight again by $w^{-\epsilon}$. We know that $r(z)$ has a meromorphic extension to $\{\mathrm{Im} z > -\delta_\epsilon\}$ for some $\delta_\epsilon > 0$. We note again $r(z)$ this meromorphic extension. Let

$$\tilde{p}(z) = \tilde{p}(z, x, \partial_x) := w^\epsilon p(z) w^\epsilon.$$

This is an elliptic second order operator with analytic coefficients. We clearly have

$$r(z) \circ \tilde{p}(z) = \tilde{p}(z) \circ r(z) = \mathbb{1}, \tag{12.7}$$

first for $\mathrm{Im} z$ sufficiently large and then in $\{\mathrm{Im} z > -\delta_\epsilon\}$ by meromorphic extension. Let $K_z(x, x')$ the distribution kernel of $r(z)$. We have :

$$\tilde{p}(z)(x, \partial_x) K_z(x, x') = \delta(x, x'),\ z \in \Omega,$$

$$\tilde{p}(z)^t(x', \partial_{x'}) K_z(x, x') = \delta(x, x'),\ z \in \Omega,$$

where $\tilde{p}(z)^t$ is the transpose of $\tilde{p}(z)$, and is also elliptic with analytic coefficients. By the Morrey-Nirenberg theorem [22, Thm. 7.5.1], $\tilde{p}(z)$ and $\tilde{p}(z)^t$ are *analytic hypo-elliptic*, from which we obtain that $K_z(x, x')$ is analytic in x, x' outside the diagonal, for $\{\mathrm{Imz} > -\delta_\epsilon\}$.

Recall from [9] that there exists $\delta_r > 0$ such that for all $\eta \in C_0^\infty((r_- + \delta_r, r_+ - \delta_r)$ $\eta p^{-1}(z)\eta$ has no poles in $\{\mathrm{Im} z > -\delta_0\}$ for some $\delta_0 > 0$. Let now $z_0 \in \{\mathrm{Im} z > -\delta_0\}$ be a possible pole of $r(z)$. We write:

$$r(z) = \sum_{i=1}^{N} P_j (z - z_0)^{-j} + H(z),$$

where P_j are finite rank operators and $H(z)$ is holomorphic close to z_0 and $P_N \neq 0$. We want to show that all the P_j are zero. Clearly it is sufficient to show that $P_N = 0$.

We have

$$P_N = \frac{1}{2\mathrm{i}\pi}\oint_\gamma (z - z_0)^{N-1} r(z) dz,$$

which shows that the kernel $P_N(x, x')$ of P_N is analytic outside the diagonal. But as $\eta p(z)^{-1}\eta$ has no poles, we necessarily have $P_N(x, x') = 0$ for $x \neq x'$, $x, x' \in \operatorname{supp} \eta$. By analytic continuation we therefore have $P_N(x, x') = 0$ for $x \neq x'$. We then have

$$\begin{aligned}\tilde{p}(z_0)P_N = & \tfrac{1}{2\mathrm{i}\pi}\oint_\gamma \tilde{p}(z_0)(z - z_0)^{N-1} r(z) dz \\ = & \tfrac{1}{2\mathrm{i}\pi}\oint_\gamma (z - z_0)^{N-1}(\tilde{p}(z_0) - \tilde{p}(z)) r(z) dz \\ & + \tfrac{1}{2\mathrm{i}\pi}\oint_\gamma (z - z_0)^{N-1}\tilde{p}(z) r(z) dz.\end{aligned}$$

As $\tilde{p}(z) - \tilde{p}(z_0) = (z - z_0)T(z)$ with $T(z)$ holomorphic close to z_0, the first term is zero, the second is zero because $\tilde{p}(z)r(z) = \mathbb{1}$. It follows that $\tilde{p}(z_0)P_N = 0$.

Let us show that this implies $P_N = 0$. Let $u \in L^2(\mathbb{R} \times S^2)$ with compact support. As the distribution kernel of P_N is supported on the diagonal, $v = P_N u$ has also compact support and $\tilde{p}(z_0)v = 0$. Again by analytic hypo-ellipticity of $\tilde{p}(z_0)$, we obtain that v is analytic with compact support, thus $v = 0$. By a density argument we obtain $P_N = 0$. This completes the proof of *i)*.

Let us now prove *ii)*. By [16, Proposition 9.3] we know that $\mathcal{T}_\pm \cap \mathbb{R}\backslash\{0\} = \emptyset$. By Corollary 5.12 it is sufficient to show that 0 is not a resonance of $w^{-\epsilon}p_\pm^{-1}(z)w^{-\epsilon}$. We treat the + case, the − case being analogous. Suppose that 0 is a resonance. First note that $p_+(0)$ is an elliptic operator with $p_+(0) \gtrsim n^2 w^{-2}$. In particular $w^\epsilon p_+(0)w^\epsilon v = 0$ implies $v = 0$. Let $r(z) = w^{-\epsilon}p_+^{-1}(z)w^{-\epsilon}$. Suppose that $r(z)$ has a pole at $z = 0$:

$$r(z) = \sum_{j=1}^{N} \frac{P_j}{z^j} + H(z), \quad P_N \neq 0.$$

Here P_j are of finite rank and $H(z)$ is holomorphic. Let $u \in \mathcal{H}$ such that $P_N u \neq 0$. We have

$$z^N u = \sum_{j=1}^{N} z^{N-j} w^\epsilon p_+(z) w^\epsilon P_j u + z^N w^\epsilon p_+(z) w^\epsilon H(z) u.$$

In the limit $z \to 0$ we obtain $w^\epsilon p_+(z)w^\epsilon P_N u = 0$ and thus $P_N u = 0$ which is a contradiction. This completes the proof of the proposition. $\square$

12.3. **Proof of Thm. 11.2.** We will apply here the results of Sect. 9. First note that in our new setting (i.e. after rotation) we have to consider

$$\begin{aligned} h_{-\infty} &:= -\ell^2 - \partial_x^2 + \frac{\Delta_r}{r^2+a^2}P + \frac{\Delta_r m^2}{\lambda^2(r^2+a^2)}, \\ k_{-\infty} &:= \ell, \\ h_{+\infty} &:= -\partial_x^2 + \frac{\Delta_r}{r^2+a^2}P + \frac{\Delta_r m^2}{\lambda^2(r^2+a^2)}, \\ k_{+\infty} &:= 0, \\ \ell &:= \left(\frac{a}{r_+^2+a^2} - \frac{a}{r_-^2+a^2}\right)n. \end{aligned}$$

We associate to these operators the operators $H_{\pm\infty}$, $\dot{H}_{\pm\infty}$ and spaces $\mathcal{E}_{\pm\infty}$, $\dot{\mathcal{E}}_{\pm\infty}$ as in Sect. 2. Let $\mathcal{T}_{\pm\infty}$ be the set of singular points of $\dot{H}_{\pm\infty}$. We have :

Lemma 12.2. *For $n \neq 0$ we have $\mathcal{T}_{\pm\infty} = \emptyset$.*

Proof. As $\dot{H}_{\pm\infty}$ is selfadjoint we can use the Kato theory of $H-$ smoothness. The proof for the absence of real resonances is analogous to the proof of Prop. 12.1 *ii)*, we omit the details. □

12.4. **Proof of Thm. 11.2.** We first consider the case $n \neq 0$. By Prop. 12.1 we know that $\sigma_{pp}^{\mathbb{C}}(\dot{H}) = \mathcal{S} = \mathcal{T}_{\pm\infty} = \emptyset$. Thus $\mathbb{1} = \mathbb{1}_{\mathbb{R}}(\dot{H})$ is an admissible energy cut-off. Using in addition that $e^{-it\dot{H}}$, $e^{-it\dot{H}_{\pm\infty}}$ are uniformly bounded, the theorem follows from Thm. 9.5. In the case $n = 0$ all operators are selfadjoint. This case then follows from [19], we omit the details. □

12.5. **Proof of Thm. 11.3.** We first write the comparison dynamics which we obtain after rotation:

$$\begin{aligned} h_r &= -\partial_x^2,\ h_l = -\partial_x^2 - \ell^2, \\ k_r &= 0,\ k_l = \ell. \end{aligned}$$

We associate to these operators the natural homogeneous energy spaces $\dot{\mathcal{E}}_{l/r}$. Let

$$\dot{H}_r = \begin{pmatrix} 0 & \mathbb{1} \\ h_r & 2k_r \end{pmatrix}, \quad \dot{H}_l = \begin{pmatrix} 0 & \mathbb{1} \\ h_l & 2k_l \end{pmatrix}.$$

We now further analyze the energy spaces. Note that

$$\begin{aligned} \dot{\mathcal{E}}_l &= \Phi(\ell)(H^1(\mathbb{R}); L^2(S^2)) \oplus L^2(\mathbb{R}\times S^2)), \\ \dot{\mathcal{E}}_r &= H^1(\mathbb{R}; L^2(S^2)) \oplus L^2(\mathbb{R}\times S^2). \end{aligned}$$

We will need the following subspaces

$$\begin{aligned} \dot{\mathcal{E}}_l^L &= \{(u_0,u_1)\in\dot{\mathcal{E}}_l;\ u_1 - i\ell u_0 \in L^1(\mathbb{R}; L^2(S^2)), \int (u_1 - i\ell u_0)(x,\omega)dx = 0 \quad \text{a.e. in} \quad \omega\}, \\ \dot{\mathcal{E}}_r^L &= \{(u_0,u_1)\in\dot{\mathcal{E}}_r;\ u_1 \in L^1(\mathbb{R}; L^2(S^2)), \int u_1(x,\omega)dx = 0 \quad \text{a.e. in} \quad \omega\}. \end{aligned}$$

We define the spaces of *incoming /outgoing* initial data:

$$\begin{array}{rcl}\dot{\mathcal{E}}_l^{\rm in} &=& \{u\in\dot{\mathcal{E}}_l^L;\ u_1=\partial_x u_0+i\ell u_0\},\\ \dot{\mathcal{E}}_l^{\rm out} &=& \{u\in\dot{\mathcal{E}}_l^L;\ u_1=-\partial_x u_0+i\ell u_0\},\\ \dot{\mathcal{E}}_r^{\rm in} &=& \{u\in\dot{\mathcal{E}}_r^L;\ u_1=\partial_x u_0\},\\ \dot{\mathcal{E}}_r^{\rm out} &=& \{u\in\dot{\mathcal{E}}_r^L;\ u_1=-\partial_x u_0\}.\end{array}$$

If $(u_0,u_1)\in\dot{\mathcal{E}}_l^{\rm in}$, then the solution of (11.7) is given by

$$u_0(t,x,\omega)=e^{i\ell t}u_0(x+t,\omega),$$

which is clearly incoming. We have the following

Lemma 12.3. *We have*

$$\dot{\mathcal{E}}_l^L=\dot{\mathcal{E}}_l^{\rm in}\oplus\dot{\mathcal{E}}_l^{\rm out},\quad \dot{\mathcal{E}}_r^L=\dot{\mathcal{E}}_r^{\rm in}\oplus\dot{\mathcal{E}}_r^{\rm out}.$$

Proof. We only show the lemma for $\dot{\mathcal{E}}_l^L$, $\dot{\mathcal{E}}_r^L$ being the special case $\ell=0$. We define for $u=(u_0,u_1)\in\dot{\mathcal{E}}_l^L$:

$$\begin{array}{rcl}u_0^{\rm in} &=& \frac12\int_x^\infty(-\partial_x u_0-(u_1-i\ell u_0))(\tau,\omega)d\tau,\\ u_1^{\rm in} &=& \frac12(u_1-i\ell u_0+\partial_x u_0)+\frac{i\ell}{2}\int_x^\infty(-\partial_x u_0-(u_1-i\ell u_0))(\tau,\omega)d\tau,\\ u_0^{\rm out} &=& \frac12\int_{-\infty}^x(\partial_x u_0-(u_1-i\ell u_0))(\tau,\omega)d\tau,\\ u_1^{\rm out} &=& \frac12(u_1-i\ell u_0-\partial_x u_0)+\frac{i\ell}{2}\int_{-\infty}^x(\partial_x u_0-(u_1-i\ell u_0))(\tau,\omega)d\tau,\\ u^{\rm in/out} &=& (u_0^{\rm in/out},u_1^{\rm in/out}).\end{array}\tag{12.8}$$

It is easy to check that

$$u=u^{\rm in}+u^{\rm out},\ u^{\rm int/out}\in\dot{\mathcal{E}}_\ell^{\rm in/out},$$

which shows that $\dot{\mathcal{E}}_\ell^{\rm in}+\dot{\mathcal{E}}_\ell^{\rm out}=\dot{\mathcal{E}}_\ell$. Next if $v\in\dot{\mathcal{E}}_\ell^{\rm in}\cap\dot{\mathcal{E}}_\ell^{\rm out}$ we have $\partial_x v_0=0$, $v_0\in L^2$ hence $v_0=0$, hence $v_1=0$. $\square$

Remark 12.4. *If* $(u_0,u_1)\in\dot{\mathcal{E}}_l^L$ *or* $(u_0,u_1)\in\dot{\mathcal{E}}_r^L$, $\operatorname{supp}u_0$, $\operatorname{supp}u_1\subset]R_1,R_2[\times S^2$, *then we have*

$$\operatorname{supp}u_{0,1}^{\rm in}\subset]-\infty,R_2[\times S^2,\quad \operatorname{supp}u_{0,1}^{\rm out}\subset]R_1,\infty[\times S^2.\tag{12.9}$$

The spaces $\mathcal{E}^q_{l/r}$, $\mathcal{E}^{fin}_{l/r}$ are defined as before but starting with our now slightly modified operators (due to the rotation). Let

$$\mathcal{D}^{fin}_{r/l}=C_0^\infty(\mathbb{R}\times S^2)\times C_0^\infty(\mathbb{R}\times S^2)\cap\mathcal{E}^{fin}_{r/l}\cap\dot{\mathcal{E}}^L_{r/l}.$$

Lemma 12.5. $\mathcal{D}^{fin}_{r/l}$ *is dense in* $\mathcal{E}^{fin}_{r/l}$.

Proof. We prove the lemma in two steps.

- $C_0^\infty(\mathbb{R}\times S^2)\times C_0^\infty(\mathbb{R}\times S^2)\cap\mathcal{E}^{fin}_{r/l}$ is dense in $\mathcal{E}^{fin}_{r/l}$. This follows easily from the usual regularization procedures.

$$\overline{C_0^\infty(\mathbb{R}\times S^2)\times C_0^\infty(\mathbb{R}\times S^2)\cap\mathcal{E}^{fin}_{r/l}\cap\dot{\mathcal{E}}^L_{r/l}}\text{ is dense in } C_0^\infty(\mathbb{R}\times S^2)\times C_0^\infty(\mathbb{R}\times S^2)\cap\mathcal{E}^{fin}_{r/l}.$$

We can clearly replace $\mathcal{E}^{fin}_{r/l}$ by $\mathcal{E}^q_{r/l}$ in the statement. We only treat the $l-$ case. Let

$$u=(u_0,u_1)\in C_0^\infty(\mathbb{R}\times S^2)\times C_0^\infty(\mathbb{R}\times S^2)\cap\mathcal{E}^q_l.$$

We will consider u as a function of x alone. We put

$$v=\Phi(-\ell)u.$$

Let $\psi\in C_0^\infty(\mathbb{R})$, $\psi\geq 0$, $\psi=1$ in a neighborhood of zero and $\int\psi(x)dx=1$. We put

$$v_0^n=v_0,\quad v_1^n=v_1-n^{-1}\psi(n^{-1}x)\int v_1(x)dx,$$

so that $\int v_1^n(x)dx=0$. We then estimate

$$\|v_1-v_1^n\|_{L^2}\leq n^{-\frac12}\|v_1\|_{L^1}\|n^{-\frac12}\psi(n^{-1}\cdot)\|_{L^2}\leq Cn^{-\frac12}\|v_1\|_{L^1}\to 0,$$

which completes the proof.

□

We need an additional

Lemma 12.6. *There exists $C>0$ such that*

$$\|i_{r/l}u\|_{\dot{\mathcal{E}}_{r/l}}\leq C\|u\|_{\dot{\mathcal{E}}},\ u\in\dot{\mathcal{E}}.$$

Proof. We have:

$$\begin{aligned}\|i_ru\|^2_{\dot{\mathcal{E}}_r}&=\|i_ru_1\|^2_{\mathcal{H}}+(i_rh_{+\infty}i_ru_0|u_0)\\&\lesssim\|(u_1-ku_0)\|^2_{\mathcal{H}}+(i_r(h_{+\infty}+k^2)i_ru_0|u_0)\\&\lesssim\|(u_1-ku_0)\|^2_{\mathcal{H}}+(h_0u_0|u_0)=\|u\|^2_{\dot{\mathcal{E}}}.\end{aligned}$$

Now recall that

$$\tilde{h}_{-\infty}=-\partial_x^2+\frac{\Delta_r}{r^2+a^2}P+m^2\Delta_r.$$

We then estimate

$$\begin{aligned}\|i_lu\|^2_{\dot{\mathcal{E}}_l}&=\|i_l(u_1-\ell u_0)\|^2+(i_l\tilde{h}_{-\infty}i_lu_0|u_0)\\&\lesssim\|i_l(u_1-ku_0)\|^2_{\mathcal{H}}+(i_l(\tilde{h}_{-\infty}+(k-\ell)^2)i_lu_0|u_0)\\&\lesssim\|(u_1-ku_0)\|^2_{\mathcal{H}}+(h_0u_0|u_0)=\|u\|^2_{\dot{\mathcal{E}}}.\end{aligned}$$

□

Proof of Thm. 11.3. We first show for $u\in\mathcal{E}^{fin}_{r/l}$ the existence of the limit

$$\tilde{W}_{r/l}u=\lim_{t\to\infty}e^{it\dot{H}_{\pm\infty}}i_{r/l}e^{-it\dot{H}_{r/l}}u$$

in $\dot{\mathcal{E}}_{\pm\infty}$. Let $u\in\oplus_{|q|\leq Q}\mathcal{E}^q$. Using the estimate

$$\|e^{it\dot{H}_{\pm\infty}}i_{r/l}e^{-it\dot{H}_{r/l}}u\|_{\dot{\mathcal{E}}_{\pm\infty}}\leq C(Q)\|u\|_{\dot{\mathcal{E}}_{r/l}}$$

as well as the spherical symmetry of the problem, it is sufficient to show for all $|q| \le Q$ the existence of the limits

$$\lim_{t\to\infty} e^{it\dot{H}^q_{\pm\infty}} i_{r/l} e^{-it\dot{H}^q_{r/l}} u^q,$$

where $u^q \in \mathcal{E}^q$ and $\dot{H}^q_{\pm\infty}$ resp. $\dot{H}^q_{r/l}$ are the restrictions of $\dot{H}_{\pm\infty}$ resp. $\dot{H}^l_{r/l}$ to $\dot{\mathcal{E}}^q_{r/l}$. The existence of this limit follows from standard arguments using the exponential decay of Δ_r at $\pm\infty$. Using Thm. 11.2 we obtain the existence of the limit

$$\lim_{t\to\infty} e^{it\dot{H}} i^2_{r/l} e^{-it\dot{H}_{r/l}} u = W_{r/l} u.$$

We now want to show that there exists $C > 0$ such that for all $u \in \mathcal{E}^{fin}_{r/l}$

$$\|W_{r/l} u\|_{\dot{\mathcal{E}}} \le C\|u\|_{\dot{\mathcal{E}}_{r/l}}. \tag{12.10}$$

We first consider W_l. By Lemma 12.5 we can suppose $(u_0, u_1) \in \mathcal{D}^{fin}_l$. Let $\mathrm{supp}\, u_0$, $\mathrm{supp}\, u_1 \subset (R_1, R_2)$. We decompose (u_0, u_1) in incoming and outgoing solutions according to the discussion at the beginning of this subsection :

$$u_0 = u^{\mathrm{in}}_{0,l} + u^{\mathrm{out}}_{0,l}, \quad u_1 = u^{\mathrm{in}}_{0,l} + u^{\mathrm{out}}_{0,l}.$$

By Remark 12.4 we have

$$\begin{aligned} \mathrm{supp}\, u^{\mathrm{in}}_{0,l},\ \mathrm{supp}\, u^{\mathrm{in}}_{1,l} &\subset]-\infty, R_2[\times S^2, \\ \mathrm{supp}\, u^{\mathrm{out}}_{0,l},\ \mathrm{supp}\, u^{\mathrm{out}}_{1,l} &\subset]R_1, \infty[\times S^2. \end{aligned}$$

Let $u^{\mathrm{in}}_l = (u^{\mathrm{in}}_{0,l}, u^{\mathrm{in}}_{1,l})$, $u^{\mathrm{out}}_l = (u^{\mathrm{out}}_{0,l}, u^{\mathrm{out}}_{1,l})$. We have

$$W_l u^{\mathrm{out}}_l = 0,$$

because

$$i^2_l e^{-it\dot{H}_l} u^{\mathrm{out}}_l = 0$$

for t sufficiently large. We have

$$\mathrm{supp}\, e^{-it\dot{H}_l} u^{\mathrm{in}}_l \subset (]-\infty, R_2 - t[\times S^2) \times (]-\infty, R_2 - t[\times S^2).$$

We then estimate for t large

$$\begin{aligned} \|e^{it\dot{H}} i^2_l e^{-it\dot{H}_l} u^{\mathrm{in}}\|_{\dot{\mathcal{E}}} &\lesssim \|i^2_l e^{-it\dot{H}_l} u^{\mathrm{in}}\|_{\dot{\mathcal{E}}} \\ &\lesssim \|u^{\mathrm{in}}\|^2_{\dot{\mathcal{E}}_l} + \left(\left(\frac{\Delta_r}{r^2+a^2} P + \Delta_r m^2\right)\left(e^{-it\dot{H}_l} u^{\mathrm{in}}\right)_0 \Big| \left(e^{-it\dot{H}_l} u^{\mathrm{in}}\right)_0\right) \\ &\lesssim \|u^{\mathrm{in}}\|^2_{\dot{\mathcal{E}}_l} + e^{-\kappa_- t}(Q+1)\|u^{\mathrm{in}}\|^2_{\mathcal{H}} \\ &\to \|u^{\mathrm{in}}\|^2_{\dot{\mathcal{E}}_l}, \quad t \to \infty. \end{aligned}$$

It follows

$$\|W_l u\|_{\dot{\mathcal{E}}} \le C\|u\|_{\dot{\mathcal{E}}_l},$$

which is the required estimate. The proof for W_r is analogous. Part *ii)* is shown in the same way. The required estimate follows from Lemma 12.6. □

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Département de Mathématiques, Université de Cergy-Pontoise, 95302 Cergy-Pontoise Cedex, France

E-mail address: `Vladimir.Georgescu@math.cnrs.fr`

Département de Mathématiques, Université de Paris XI, 91405 Orsay Cedex France

E-mail address: `christian.gerard@math.u-psud.fr`

Université de Grenoble 1, Institut Fourier, UMR 5582 CNRS, BP 74 38402 Saint-Martin d'Hères France

E-mail address: `Dietrich.Hafner@ujf-grenoble.fr`